hep-th/9609124
DESY 96-196
IHES/P/96/47
September 1996

WZW FUSION RINGS IN THE LIMIT OF INFINITE LEVEL

Jürgen Fuchs [£]

DESY
Notkestraße 85
D – 22603 Hamburg

and

Christoph Schweigert

IHES
35, Route de Chartres
F – 91440 Bures-sur-Yvette

Abstract.
We show that the WZW fusion rings at finite levels form a projective system with respect to the partial ordering provided by divisibility of the height, i.e. the level shifted by a constant. From this projective system we obtain WZW fusion rings in the limit of infinite level. This projective limit constitutes a mathematically well-defined prescription for the 'classical limit' of WZW theories which replaces the naive idea of 'sending the level to infinity'. The projective limit can be endowed with a natural topology, which plays an important rôle for studying its structure. The representation theory of the limit can be worked out by considering the associated fusion algebra; this way we obtain in particular an analogue of the Verlinde formula.

[£] Heisenberg fellow

1 Fusion rings

Fusion rings constitute a mathematical structure which emerges in various contexts, for instance in the analysis of the superselection rules of two-dimensional quantum field theories; they describe in particular the basis independent contents of the operator product algebra of two-dimensional conformal field theories (for a review see [1]). By definition, a *fusion ring* $\mathcal{R}$ is a unital commutative associative ring over the integers $\mathbb{Z}$ which possesses the following properties: there is a distinguished basis $\mathcal{B} = \{\varphi_a\}$ which contains the unit and in which the structure constants ${\mathcal{N}_{a,b}}^c$ are non-negative integers, and the evaluation at the unit induces an involutive automorphism, called the conjugation of $\mathcal{R}$. A fusion ring is referred to as *rational* iff it is finite-dimensional. A rational fusion ring is called *modular* iff the matrix S that diagonalizes simultaneously all fusion matrices $\mathcal{N}_a$ (i.e. the matrices with entries ${(\mathcal{N}_a)_b}^c = {\mathcal{N}_{a,b}}^c$) is symmetric and together with an appropriate diagonal matrix T generates a unitary representation of $\mathrm{SL}(2,\mathbb{Z})$ (see e.g. [2, 1]).

In this paper we consider the fusion rings of (chiral, unitary) WZW theories. A WZW theory is a conformal field theory whose chiral symmetry algebra is the semidirect sum of the Virasoro algebra with an untwisted affine Kac–Moody algebra $\mathfrak{g}$, with the level $k^\vee$ of the latter a fixed non-negative integer. To any untwisted affine Kac–Moody algebra $\mathfrak{g}$ we can thus associate a family of fusion rings, parametrized by the level $k^\vee$. The issue that we address in this paper is to construct an analogue of the WZW fusion ring for infinite level, which is achieved by giving a prescription for 'sending the level to infinity' in an unambiguous manner.

In view of the Lagrangian realization of WZW theories as sigma models, this procedure may be regarded as taking the 'classical limit' of WZW theories. Performing a classical limit of a parametrized family of quantum field theories is a rather common concept in the path integral formulation of quantum theories; it simply corresponds to sending Planck's constant to zero, and hence provides a kind of correspondence principle. In the Lagrangian description of WZW theories as principal sigma models with Wess–Zumino terms, the rôle of Planck's constant is played by the inverse of the level $k^\vee$ of the underlying affine Lie algebra $\mathfrak{g}$. However, it is known that the path integral of a WZW sigma model strictly makes sense only if the level $k^\vee$ is an integer. In contrast to the path integral description, in the framework of algebraic approaches to quantum theory so far almost no attempts have been made to investigate limits of quantum field theories. In this paper we address this issue for the case of WZW theories. Now in an algebraic treatment of WZW theories the integrality requirement just mentioned is immediately manifest. Namely, one observes that the structure of the theory depends sensitively on the value $k^\vee$ of the level. For non-negative integral $k^\vee$ the state space is a direct sum of unitary irreducible highest weight modules of the algebra $\mathfrak{g}$, but its structure changes quite irregularly when going from $k^\vee$ to $k^\vee + 1$; at intermediate, non-integral, values of the level there do not even exist any unitarizable highest weight representations.

These observations indicate that it is a rather delicate issue to define what is meant by the classical limit of a WZW theory, and it seems mandatory to perform this limit in a manner in which the level $k^\vee$ is manifestly kept integral (actually, treating the level formally as a continuous variable is potentially ambiguous even in situations where one deals with expressions which superficially make sense also at non-integral level). It must also be noted that a priori it is by no means clear whether the so obtained limit will be identical with or at least closely resemble the structures which originally served to define the quantum theory in terms of some

classical field theory; in the case of WZW fusion rings, this underlying classical structure is the representation ring of the finite-dimensional simple Lie algebra $\bar{\mathfrak{g}}$ that is generated by the zero modes of the affine Lie algebra $\mathfrak{g}$. Indeed, it seems to be a quite generic feature of quantum theory that the classical limit does not simply reproduce the classical structure one started with. (Compare for instance the fact that in the path integral formulation of quantum field theory the classical paths are typically of measure zero in the space of all paths that contribute to the path integral. Similar phenomena also show up when the continuum limit of a lattice theory is constructed as a projective limit; see e.g. [3, 4].) However, it is still reasonable to expect that the original classical structure is, in a suitable manner, contained in the classical limit; as we will see, this is indeed the case for our construction.

The desire of being able to perform a limit in which the level tends to infinity stems in part from the fact that WZW theories and their fusion rings can be used to define a regularization of various systems (such as two-dimensional gauge theories or the Ponzano–Regge theory of simplicial three-dimensional gravity), with the unregularized system corresponding, loosely speaking, to the classical theory. As removing the regulator is always a subtle issue, it is mandatory that the limit of the regularized theory is performed in a well-defined, controllable manner, which, in addition, should preserve as much of the structure as possible.

The basic idea which underlies our construction of the limit of WZW fusion rings is to interpret the collection of WZW fusion rings as a category $\mathcal{F}us(\mathfrak{g})$ within the category of all commutative rings and identify inside this category a projective system. By a standard category theoretic construction we can then obtain the limit (also known as the projective limit) of this projective system. The partial ordering underlying the projective system is based on a divisibility property of the parameter $k^\vee + g^\vee$ that together with the choice of the horizontal subalgebra $\bar{\mathfrak{g}}$ characterizes the WZW theory ($g^\vee$ denotes the dual Coxeter number of $\bar{\mathfrak{g}}$; the sum $k^\vee + g^\vee$ is called the *height*). In contrast, in the literature often a purely formal prescription '$k^\vee \to \infty$' is referred to as the classical limit of WZW theories. In that terminology it is implicit that the standard ordering on the set of levels is chosen to give it the structure of a directed set. Now the projective limit is associated to a projective system as a whole, not just to the collection of objects that appear in the system; in particular it depends on the underlying directed set and hence on the choice of partial ordering. Our considerations show, as a by-product, that it is not possible to associate to the standard ordering any well-defined limit of the fusion rings.

The rest of this paper is organized as follows. We start in subsection 2.1 by introducing the category $\mathcal{F}us(\mathfrak{g})$ of WZW fusion rings associated to an untwisted affine Lie algebra $\mathfrak{g}$; in subsection 2.2 conditions for the existence of non-trivial morphisms of this category are obtained. In subsection 2.3 we define the projective system, and in the remainder of section 2 we check that the morphisms introduced by this definition possess the required properties. The projective limit of the so obtained projective system is a unital commutative associative ring of countably infinite dimension. This ring ${}^{(\infty)}\mathcal{R}$ is constructed in section 3; there we also gather some basic properties of ${}^{(\infty)}\mathcal{R}$ and introduce a natural topology on ${}^{(\infty)}\mathcal{R}$. In section 4 a concrete description of a distinguished basis ${}^{(\infty)}\mathcal{B}$ for the projective limit is obtained. This basis is similar to the distinguished bases of the fusion rings at finite level; in order to show that ${}^{(\infty)}\mathcal{R}$ is indeed generated by ${}^{(\infty)}\mathcal{B}$, the topology on ${}^{(\infty)}\mathcal{R}$ plays an essential rôle. In section 5 we demonstrate that the representation ring of the horizontal subalgebra $\bar{\mathfrak{g}} \subset \mathfrak{g}$ is contained in the projective limit ${}^{(\infty)}\mathcal{R}$ as a proper subring. In the final section 6 we study the representation

theory of ${}^{(\infty)}\mathcal{R}$, respectively of the associated fusion algebra over $\mathbb{C}$. In particular, we determine all irreducible representations and show that ${}^{(\infty)}\mathcal{R}$ possesses a property which is the topological analogue of semi-simplicity, namely that any continuous ${}^{(\infty)}\mathcal{R}$-module is the closure of a direct sum of irreducible modules. To obtain these results it is again crucial to treat the projective limit as a topological space. Finally, we establish an analogue of the Verlinde formula for ${}^{(\infty)}\mathcal{R}$.

2 The projective system of WZW fusion rings

2.1 WZW fusion rings

The primary fields of a unitary WZW theory are labelled by integrable highest weights of the relevant affine Lie algebra $\mathfrak{g}$, or what is the same, by the value $k^\vee$ of the level and by dominant integral weights Λ of $\bar{\mathfrak{g}}$ (the horizontal subalgebra of $\mathfrak{g}$) whose inner product with the highest coroot of $\bar{\mathfrak{g}}$ is not larger than $k^\vee$. We denote by $g^\vee$ the dual Coxeter number of the simple Lie algebra $\bar{\mathfrak{g}}$ and define

$$I := \{i \in \mathbb{Z} \mid i \geq g^\vee\}\,. \tag{2.1}$$

Thus I is the set of possible values of the *height* $h \equiv k^\vee + g^\vee$ of the WZW theory based on $\mathfrak{g}$. For any $h \in I$ the fusion rules of a WZW theory at height h define a modular fusion ring, with the elements of the distinguished basis corresponding to the primary fields. We denote this ring by ${}^{(h)}\mathcal{R}$ and its distinguished basis by ${}^{(h)}\mathcal{B}$, and the corresponding generators of $\mathrm{SL}(2,\mathbb{Z})$ by ${}^{(h)}S$ and ${}^{(h)}T$.

The distinguished basis ${}^{(h)}\mathcal{B}$ of the ring ${}^{(h)}\mathcal{R}$ can be labelled as

$${}^{(h)}\mathcal{B} = \{{}^{(h)}\varphi_a \,|\, a \in {}^{(h)}P\} \tag{2.2}$$

by the set

$${}^{(h)}P := \{a \in \overline{L}^{\mathrm{w}} \mid a^i \geq 1 \text{ for } i = 1,2,\ldots,r;\ (a,\theta^\vee) < h\} \tag{2.3}$$

of integral weights in the interior of (the horizontal projection of) the fundamental Weyl chamber of $\mathfrak{g}$ at *level* h; here r, $\theta^\vee$ and $\overline{L}^{\mathrm{w}}$ denote the rank, the highest coroot and the weight lattice of $\bar{\mathfrak{g}}$, respectively. Note that from here on we use shifted $\bar{\mathfrak{g}}$-weights $a = \Lambda + \rho$, which have level $h = k^\vee + g^\vee$, in place of unshifted weights Λ which are at level $k^\vee$. Here ρ is the Weyl vector of $\bar{\mathfrak{g}}$; in particular, $a = \rho$ is the label of the unit element of ${}^{(h)}\mathcal{R}$. This convention will simplify various formulæ further on.

The ring product of ${}^{(h)}\mathcal{R}$ will be denoted by the symbol '$\star$'; thus the fusion rules are written as

$${}^{(h)}\varphi_a \star {}^{(h)}\varphi_b = \sum_{c \in {}^{(h)}P} {}^{(h)}\mathcal{N}_{a,b}^{\ \ c}\, {}^{(h)}\varphi_c\,. \tag{2.4}$$

The collection $({}^{(h)}\mathcal{R})_{h\in I}$ of such WZW fusion rings forms a category, more precisely a subcategory of the category of commutative rings, which we denote by $\mathcal{F}us(\mathfrak{g})$. The objects of $\mathcal{F}us(\mathfrak{g})$ are the rings ${}^{(h)}\mathcal{R}$, and the morphisms (arrows) are those ring homomorphisms (which are automatically unital and compatible with the conjugation) which map the basis ${}^{(h')}\mathcal{B}$ up to sign factors to ${}^{(h)}\mathcal{B}$. These are the natural requirements to be imposed on morphisms. Namely, one preserves precisely all structural properties of the fusion ring, except for the positivity of the structure constants; the latter is not an algebraic property, so that one should be prepared to give it up.

2.2 Existence of morphisms

It is not a priori clear whether the category $\mathcal{F}us(\mathfrak{g})$ as defined above has any non-trivial morphisms at all. To analyze this issue, we consider the quotients ${}^{(h)}\!S_{a,b}/{}^{(h)}\!S_{a,\rho}$ $(a,b\in{}^{(h)}\!P)$ of S-matrix elements. These are known as the (generalized) quantum dimensions, or more precisely, as the ath quantum dimension of the element ${}^{(h)}\!\varphi_b$, of the modular fusion ring ${}^{(h)}\!\mathcal{R}$. The generalized quantum dimensions furnish precisely all inequivalent irreducible representations of ${}^{(h)}\!\mathcal{R}$ [2]. We denote by

$$ {}^{(h)}\!\pi_a: \quad {}^{(h)}\!\mathcal{R}\to\mathbb{C}\,, \quad {}^{(h)}\!\varphi_b\mapsto\frac{{}^{(h)}\!S_{a,b}}{{}^{(h)}\!S_{a,\rho}} \tag{2.5}$$

the irreducible representation of ${}^{(h)}\!\mathcal{R}$ which associates to any element its ath generalized quantum dimension.

Assume now that $f:\ {}^{(h')}\!\mathcal{R}\to{}^{(h)}\!\mathcal{R}$ is a non-trivial morphism, i.e. a ring homomorphism which maps the distinguished basis ${}^{(h')}\!\mathcal{B}$ of ${}^{(h')}\!\mathcal{R}$ to the basis ${}^{(h)}\!\mathcal{B}$ of ${}^{(h)}\!\mathcal{R}$. Then the composition ${}^{(h)}\!\pi_a\circ f$ provides us with a one-dimensional, and hence irreducible, representation of ${}^{(h')}\!\mathcal{R}$, i.e. we have ${}^{(h)}\!\pi_a\circ f={}^{(h')}\!\pi_{a'}$ for some $a'\in{}^{(h')}\!P$. Let now ${}^{(h)}\!L$ denote the extension of the field $\mathbb{Q}$ of rational numbers by the quantum dimensions ${}^{(h)}\!S_{a,b}/{}^{(h)}\!S_{a,\rho}$ of all elements of ${}^{(h)}\!\mathcal{B}$; the observation just made then implies that

$$ {}^{(h)}\!L\subseteq{}^{(h')}\!L \tag{2.6}$$

(when f is surjective, one gets in fact the whole field ${}^{(h)}\!L$). As we will see, this result puts severe constraints on the existence of morphisms from ${}^{(h')}\!\mathcal{R}$ to ${}^{(h)}\!\mathcal{R}$. It follows from the Kac–Peterson formula [5] for the S-matrix that

$$ {}^{(h)}\!L\subseteq\mathbb{Q}(\zeta_{Mh})\,, \tag{2.7}$$

with $\zeta_m:=\exp(2\pi\mathrm{i}/m)$ and M the smallest positive integer for which all entries of the metric on the weight space of $\bar{\mathfrak{g}}$ are integral multiples of $1/M$ (except for $\bar{\mathfrak{g}}=A_r$ where $M=r+1$, M satisfies $M\le 4$). The inclusion (2.6) therefore implies that ${}^{(h)}\!L$ lies in the intersection $\mathbb{Q}(\zeta_{Mh})\cap\mathbb{Q}(\zeta_{Mh'})=\mathbb{Q}(\zeta_{M\,\mathrm{lcd}(h,h')})$, and that this intersection is strictly larger than $\mathbb{Q}$ unless ${}^{(h)}\!L=\mathbb{Q}$. Here $\mathrm{lcd}(m,n)$ stands for the largest common divisor of m and n. In the specific case that h and h' are coprime, $\mathrm{lcd}(h,h')=1$, it follows that

$$ {}^{(h)}\!L\ \subseteq\ {}^{(h')}\!L\cap\mathbb{Q}(\zeta_{Mh})\ \subseteq\ \mathbb{Q}(\zeta_M)\,. \tag{2.8}$$

Now typically the field ${}^{(h)}\!L$ is quite a bit smaller than $\mathbb{Q}(\zeta_{Mh})$, i.e. the inequality (2.7) is not saturated (e.g. if the ring is self-conjugate, ${}^{(h)}\!L$ is already contained in the maximal real subfield of $\mathbb{Q}(\zeta_{Mh})$); nevertheless, inspection shows that the requirement ${}^{(h)}\!L\subseteq\mathbb{Q}(\zeta_M)$ is fulfilled only in very few cases (for instance, for $\bar{\mathfrak{g}}$ of type B_{2n}, C_r, D_{2n}, E_7, E_8 or F_4, one has $M\le 2$ so that the requirement is just ${}^{(h)}\!L=\mathbb{Q}$). In addition, the main quantum dimensions ${}^{(h)}\!S_{a,\rho}/{}^{(h)}\!S_{\rho,\rho}$ lie in fact in $\mathbb{Q}(\zeta_{2h})$, and hence the above requirement would restrict them to lie in $\mathbb{Q}(\zeta_{2h})\cap\mathbb{Q}(\zeta_M)=\mathbb{Q}(\zeta_{\mathrm{lcd}(2h,M)})$, and thus to be rational whenever $2h$ and M are coprime.

It follows that for almost all pairs h, h' of coprime heights there cannot exist any morphism from ${}^{(h')}\!\mathcal{R}$ to ${}^{(h)}\!\mathcal{R}$. The same arguments also show that the existence of non-trivial morphisms becomes the more probable the larger the value of $\mathrm{lcd}(h,h')$ is. The most favourable situation is when h' is a multiple of h; in the next section we will show that in this case a whole family of morphisms from ${}^{(h')}\!\mathcal{R}$ to ${}^{(h)}\!\mathcal{R}$ (with $h\in I$ arbitrary) can be constructed in a natural way.

The considerations above indicate in particular that the naive way of taking the limit '$k \to \infty$' with the standard ordering on the set I cannot correspond to any well-defined limit of the WZW fusion rings. In contrast, as we will show, when replacing the standard ordering by a suitable partial ordering, a limit can indeed be constructed, namely as the projective limit of a projective system that is associated to that partial ordering.

Let us also mention that the required ring homomorphism property implies that any morphism of $\mathcal{F}us(\mathfrak{g})$ maps simple currents to simple currents. (By definition, simple currents are those elements φ_a of the distinguished basis ${}^{(h)}\mathcal{B}$ which have inverses in the fusion ring; they satisfy $\sum_c \mathcal{N}_{a,b}{}^c = 1$ for all b. Such elements are sometimes also called units of the ring, not to be confused with the unit element of the fusion ring.)

2.3 The projective system

On the set I (2.1) of heights one can define a partial ordering '$\preceq$' by

$$i \preceq j \;:\Leftrightarrow\; i \,|\, j \,, \tag{2.9}$$

where the vertical bar stands for divisibility. For any two elements $i, i' \in I$ there then exists a $j \in I$ (for example, the smallest common multiple of i and i') such that $i \preceq j$ and $i' \preceq j$. Thus the partial ordering (2.9) endows I with the structure of a *directed set*.

We will now show that when the set I is considered as a directed set via the partial ordering (2.9), the collection $({}^{(h)}\mathcal{R})_{h\in I}$ of WZW fusion rings can be made into a *projective system*, that is, for each pair $i, j \in I$ satisfying $i \preceq j$ there exists a morphism

$$f_{j,i}: \quad {}^{(j)}\mathcal{R} \to {}^{(i)}\mathcal{R} \,, \tag{2.10}$$

such that $f_{i,i}$ is the identity for all $i \in I$ and such that for all $i, j, k \in I$ which satisfy $i \preceq j \preceq k$, the diagram

$$\begin{tikzcd} & {}^{(k)}\mathcal{R} \arrow[dl, "f_{k,j}"'] \arrow[dr, "f_{k,i}"] & \\ {}^{(j)}\mathcal{R} \arrow[rr, "f_{j,i}"'] & & {}^{(i)}\mathcal{R} \end{tikzcd} \tag{2.11}$$

commutes.

We have to construct the maps $f_{i,j}$ for all pairs i, j with $i|j$. Writing $i = h$ and $j = \ell h$ with $\ell \in \mathbb{N}$, the construction goes as follows. The horizontal projection ${}^{(h)}W$ of the affine Weyl group at height h has the structure of a semidirect product ${}^{(h)}W = \overline{W} \ltimes h\overline{L}^{\vee}$, with $\overline{W}$ the Weyl group and $\overline{L}^{\vee}$ the coroot lattice of $\bar{\mathfrak{g}}$, so that in particular ${}^{(h)}W$ is contained as a finite index subgroup in ${}^{(\ell h)}W$, the index having the value ℓ^r. Thus any orbit of ${}^{(h)}W$ decomposes into orbits of ${}^{(\ell h)}W$, and each Weyl chamber at height ℓh is the union of ℓ^r Weyl chambers at height h. As a consequence, we find that the following statement holds for the set ${}^{(\ell h)}P$ defined according to (2.3). To any $a \in {}^{(\ell h)}P$ there either exists a unique element $w_a \in {}^{(h)}W$ such that

$$a' := w_a(a) \tag{2.12}$$

belongs to the set ${}^{(h)\!}P$, or else a lies on the boundary of some affine Weyl chamber at height h. In the former case we define

$$f_{\ell h,h}({}^{(\ell h)\!}\varphi_a) := \epsilon_\ell(a) \cdot {}^{(h)\!}\varphi_{a'} \tag{2.13}$$

with $\epsilon_\ell(a) = \mathrm{sign}(w_a)$, while in the latter case we set $f_{\ell h,h}({}^{(\ell h)\!}\varphi_a) := 0$. It is convenient to include this latter case into the formula (2.13), which is achieved by setting

$$\epsilon_\ell(a) := \begin{cases} 0 & \text{if } a \text{ lies on the boundary of an} \\ & \text{affine Weyl chamber at height } h\,, \\ \mathrm{sign}(w_a) & \text{else}\,. \end{cases} \tag{2.14}$$

2.4 Proof of the morphism properties

We have to prove that $f_{i,j}$ defined this way is a ring homomorphism and that it satisfies the composition property (2.11). It is obvious from the definition that $f_{i,i} = \mathsf{id}$ (and also that $f_{i,j}$ is surjective). To show the homomorphism property, we write $f_{i,j}$ in matrix notation, and for convenience use capital letters for the fusion ring ${}^{(\ell h)\!}\mathcal{R}$ and lower case letters for the fusion ring ${}^{(h)\!}\mathcal{R}$. Thus the elements of the basis ${}^{(\ell h)\!}\mathcal{B}$ of ${}^{(\ell h)\!}\mathcal{R}$ are denoted by $\phi_A \equiv {}^{(\ell h)\!}\varphi_A$ with $A \in {}^{(\ell h)\!}P$, while for the elements of ${}^{(h)\!}\mathcal{B}$ we just write φ_a with $a \in {}^{(h)\!}P$, and we use the notation S and s for the S-matrices in place of ${}^{(\ell h)\!}S$ and ${}^{(h)\!}S$, respectively. The mapping is then defined on the preferred basis ${}^{(\ell h)\!}\mathcal{B}$ as

$$f_{\ell h,h}(\phi_A) = \sum_{b \in {}^{(h)\!}P} F_{A,b}\, \varphi_b \tag{2.15}$$

with

$$F_{A,b} \equiv {}^{(\ell h,h)\!}F_{A,b} := \epsilon_\ell(A)\, \delta_{w_A(A),b}\,, \tag{2.16}$$

and extended linearly to all of ${}^{(\ell h)\!}\mathcal{R}$. As has been established in [6], the matrix (2.16) satisfies the relations [1]

$$\mathrm{S}\, F = \ell^{r/2}\, D\, \mathrm{s}\,, \qquad F\, \mathrm{s} = \ell^{r/2}\, \mathrm{S}\, D\,, \tag{2.17}$$

with

$$D_{A,b} \equiv {}^{(\ell h,h)\!}D_{A,b} := \delta_{A,\ell b}\,. \tag{2.18}$$

Furthermore, from the Kac–Peterson formula [5] for the modular matrix S, one deduces the identity

$$\mathrm{s}_{a,b} = \ell^{r/2}\, \mathrm{S}_{\ell a,b} \tag{2.19}$$

for all $a, b \in {}^{(h)\!}P$.

[1] In [6], mappings of the type (2.13) were encountered as so-called quasi-Galois scalings. In that setting, the level of the WZW theory is not changed, while the weights A are scaled by a factor of ℓ, followed by an appropriate affine Weyl transformation to bring the weight ℓA back to the Weyl chamber ${}^{(\ell h)\!}P$ or to its boundary. Since what matters is only the relative 'size' of weights and the translation part of the Weyl group, these mappings are effectively the same as in the present setting where there is no scaling of the weights but the extension from ${}^{(\ell h)\!}W$ to ${}^{(h)\!}W$ scales the translation lattice down by a factor of ℓ.

Note that in [6] the letter P was used for the matrix (2.16) in place of F, and D was defined as the transpose of the matrix (2.18).

Combining the relations (2.16) – (2.19) and the Verlinde formula [7], we obtain for any pair $A,B\in{}^{(\ell h)\!}P$

$$
\begin{aligned}
f_{\ell h,h}(\phi_A\star\phi_B) &= \sum_{C\in{}^{(\ell h)}P} {}^{(\ell h)\!}\mathcal{N}_{A,B}^{\ \ C}\, f_{\ell h,h}(\phi_C) = \sum_{C,D\in{}^{(\ell h)}P}\ \sum_{e\in{}^{(h)}P} \frac{\mathrm{S}_{A,D}\mathrm{S}_{B,D}\mathrm{S}^*_{C,D}}{\mathrm{S}_{\rho,D}}\, F_{C,e}\,\varphi_e \\
&= \sum_{D\in{}^{(\ell h)}P}\ \sum_{e\in{}^{(h)}P} \frac{\mathrm{S}_{A,D}\mathrm{S}_{B,D}(\mathrm{S}^*F)_{D,e}}{\mathrm{S}_{\rho,D}}\,\varphi_e = \ell^{\,r/2}\cdot \sum_{D\in{}^{(\ell h)}P}\ \sum_{e\in{}^{(h)}P} \frac{\mathrm{S}_{A,D}\mathrm{S}_{B,D}(D\mathrm{s}^*)_{D,e}}{\mathrm{S}_{\rho,D}}\,\varphi_e \\
&= \ell^{\,r/2}\cdot \sum_{d,e\in{}^{(h)}P} \frac{\mathrm{S}_{A,\ell d}\mathrm{S}_{B,\ell d}\,\mathrm{s}^*_{d,e}}{\mathrm{S}_{\rho,\ell d}}\,\varphi_e = \ell^{\,r}\cdot \sum_{d,e\in{}^{(h)}P} \frac{\mathrm{S}_{A,\ell d}\mathrm{S}_{B,\ell d}\,\mathrm{s}^*_{d,e}}{\mathrm{s}_{\rho,d}}\,\varphi_e \\
&= \ell^{\,r}\cdot \sum_{d,e\in{}^{(h)}P} \frac{(\mathrm{S}D)_{A,d}(\mathrm{S}D)_{B,d}\,\mathrm{s}^*_{d,e}}{\mathrm{s}_{\rho,d}}\,\varphi_e = \sum_{d,e\in{}^{(h)}P} \frac{(F\mathrm{s})_{A,d}(F\mathrm{s})_{B,d}\,\mathrm{s}^*_{d,e}}{\mathrm{s}_{\rho,d}}\,\varphi_e \\
&= \sum_{a,b,d,e\in{}^{(h)}P} F_{A,a}F_{B,b}\,\frac{\mathrm{s}_{a,d}\,\mathrm{s}_{b,d}\,\mathrm{s}^*_{e,d}}{\mathrm{s}_{\rho,d}}\,\varphi_e = \sum_{a,b,c\in{}^{(h)}P} F_{A,a}F_{B,b}\ {}^{(h)\!}\mathcal{N}_{a,b}^{\ \ c}\,\varphi_c \\
&= \sum_{a,b\in{}^{(h)}P} F_{A,a}F_{B,b}\,\varphi_a\star\varphi_b = f_{\ell h,h}(\phi_A)\star f_{\ell h,h}(\phi_B)
\end{aligned}
\tag{2.20}
$$

Thus $f_{\ell h,h}$ is indeed a homomorphism.

As a side remark, let us mention that an analogous situation arises for the conformal field theories which describe a free boson compactified on a circle of rational radius squared. These theories are labelled by an (even) positive integer h, and for each value of h the fusion ring is just the group ring $\mathbb{Z}\mathbb{Z}_h$ of the abelian group $\mathbb{Z}_h=\mathbb{Z}/h\mathbb{Z}$. The modular S-matrix is given by ${}^{(h)\!}S_{p,q}=h^{-1/2}\exp(2\pi\mathrm{i}pq/h)$, where the labels p and q which correspond to the primary fields are integers which are conveniently considered as defined modulo h. It is straightforward to check that the identities (2.17) are again valid (with r set to 1) if one defines ${}^{(\ell h,h)\!}F_{A,b}:=\delta^{(h)}_{A,b}$ and ${}^{(\ell h,h)\!}D_{A,b}:=\delta^{(\ell h)}_{A,\ell b}$, where the superscript on the δ-symbol $\delta^{(m)}_{a,b}$ indicates that equality needs to hold only modulo m. As a consequence, this way we obtain again a projective system based on the divisibility of h (the composition property is immediate). Moreover, precisely as in the case of WZW theories, with a different partial ordering of the set $\{h\}=\mathbb{Z}_{>0}$ it is not possible to define a projective system.

2.5 Proof of the composition property

Finally, the composition property (2.11) of the homomorphisms (2.13) is equivalent to the relation

$$\sum_{B\in{}^{(\ell h)}P} {}^{(\ell\ell' h,\ell h)\!}F_{\mathsf{A},B}\ {}^{(\ell h,h)\!}F_{B,c} = {}^{(\ell\ell' h,h)\!}F_{\mathsf{A},c} \tag{2.21}$$

among the projection matrices F that involve the three different heights h, ℓh and $\ell\ell' h$. Here as before the elements of ${}^{(h)\!}P$ and ${}^{(\ell h)\!}P$ are denoted by lower case and capital letters, respectively, while for the elements of ${}^{(\ell\ell' h)\!}P$ we use sans-serif font. The relation (2.21) is in fact an

immediate consequence of the definition of the homomorphisms $f_{i,j}$. The explicit proof is not very illuminating; the reader who wishes to skip it should proceed directly to section 3.

To prove (2.21), let us first assume that the left hand side does not vanish. Then there exist unique Weyl transformations $\overline{w}_1, \overline{w}_2 \in \overline{W}$ and unique vectors $\beta_1, \beta_2 \in h\overline{L}^\vee$ in the coroot lattice scaled by h, and a unique weight $B \in {}^{(\ell h)}P$, such that

$$\overline{w}_1(\mathsf{A}) + \ell\beta_1 = B\,, \qquad \overline{w}_2(B) + \beta_2 = c\,, \tag{2.22}$$

and the left hand side of (2.21) takes the value

$$\epsilon_{\ell\ell'}(\mathsf{A})\,\epsilon_\ell(B) = \mathrm{sign}(\overline{w}_1)\,\mathrm{sign}(\overline{w}_2) = \mathrm{sign}(\overline{w}_1\overline{w}_2)\,. \tag{2.23}$$

By combining the two relations (2.22), it follows that

$$\overline{w}_2\overline{w}_1(\mathsf{A}) + \beta = c\,, \tag{2.24}$$

where $\beta = \ell\,\overline{w}_2(\beta_1) + \beta_2$. Since β is again an element of $h\overline{L}^\vee$, this means that (2.24) describes, up to sign, the mapping corresponding to the right hand side of (2.21). Further, the sign of the right hand side is then given by $\mathrm{sign}(\overline{w}_2\overline{w}_1)$ and hence equal to (2.23); thus (2.21) indeed holds.

We still have to analyze (2.21) when its left hand side vanishes. Then either the Ath row of ${}^{(\ell\ell' h,\ell h)}F$ or the cth column of ${}^{(\ell h,h)}F$ must be zero. In the former case, the weight $\mathsf{A} \in {}^{(\ell\ell' h)}P$ belongs to the boundary of some Weyl chamber with respect to ${}^{(\ell h)}W$, and thus there exist $\overline{w} \in \overline{W}$ and $\beta \in h\overline{L}^\vee$ such that $\overline{w}(\mathsf{A}) + \ell\beta = \mathsf{A}$. But this means that $\mathsf{A} \in {}^{(\ell\ell' h)}P$ also lies on the boundary of some Weyl chamber with respect to ${}^{(h)}W \supset {}^{(\ell h)}W$, and hence also the right hand side of (2.21) vanishes as required. In the second case, there are unique elements $w_1 \in {}^{(\ell h)}W$ and $w_2 \in {}^{(h)}W$ satisfying $w_1(\mathsf{A}) = B$ and $w_2(B) = B$. Because of ${}^{(\ell h)}W \subset {}^{(h)}W$, w_1 can also be considered as an element of the Weyl group ${}^{(h)}W$ at height h. By assumption, w_2 is a non-trivial element of ${}^{(h)}W$. The product $w_0 := w_1^{-1}w_2w_1 \in {}^{(h)}W$ is then non-trivial, too, and satisfies

$$w_0(\mathsf{A}) = w_1^{-1}w_2w_1(\mathsf{A}) = w_1^{-1}w_2(B) = w_1^{-1}(B) = \mathsf{A}\,. \tag{2.25}$$

Thus the weight A is invariant under a non-trivial element of ${}^{(h)}W$ and hence lies on the boundary of some Weyl chamber with respect to ${}^{(h)}W$; this implies again that the right hand side of (2.21) vanishes as required.

This concludes the proof of (2.21), and hence of the claim that $({}^{(h)}\mathcal{R})_{h\in I}$ together with the maps $f_{i,j}$ constitutes a projective system.

3 The projective limit ${}^{(\infty)}\mathcal{R}$

We are now in a position to construct the projective limit ${}^{(\infty)}\mathcal{R}$ of the projective system that we introduced in subsection 2.3.

3.1 Projective limits and coherent sequences

A projective system $({}^{(h)}\mathcal{R})_{h\in I}$ in some category $\mathcal{C}$ is said to possess a *projective limit* $(\mathcal{L}, f)$ (also called the inverse limit, or simply the limit) if there exist an object $\mathcal{L}$ as well as a family f

of morphisms $f_h\colon\ \mathcal{L} \to {}^{(h)}\mathcal{R}$ (for all $h \in I$) which satisfy the following requirements (see e.g. [8]). First, for all $h, h' \in I$ with $h \preceq h'$ the diagram

$$
\begin{tikzcd}
& \mathcal{L} \arrow[dl, "f_{h'}"'] \arrow[dr, "f_h"] & \\
{}^{(h')}\mathcal{R} \arrow[rr, "f_{h',h}"'] & & {}^{(h)}\mathcal{R}
\end{tikzcd}
\tag{3.1}
$$

commutes; and second, the following *universal property* holds: for any object $\mathcal{O}$ of the category for which a family of morphisms $g_h\colon\ \mathcal{O} \to {}^{(h)}\mathcal{R}$ $(h \in I)$ exists which possesses a property analogous to (3.1), i.e.

$$
f_{h',h} \circ g_{h'} = g_h \quad \text{for } \ h \preceq h' \,, \tag{3.2}
$$

there exists a unique morphism $g\colon\ \mathcal{O} \to \mathcal{L}$ such that the diagram

$$
\begin{tikzcd}
& \mathcal{L} \arrow[dl, "f_{h'}"'] \arrow[dr, "f_h"] & \\
{}^{(h')}\mathcal{R} \arrow[rr, "f_{h',h}"'] & & {}^{(h)}\mathcal{R} \\
& \mathcal{O} \arrow[ul, "g_{h'}"] \arrow[ur, "g_h"'] \arrow[uu, bend right=90, "g"'] &
\end{tikzcd}
\tag{3.3}
$$

commutes for all $h, h' \in I$ with $h \preceq h'$.

To be precise, in the above characterization of the projective limit $(\mathcal{L}, f)$ it is implicitly assumed that $\mathcal{L}$ is an object in $\mathcal{C}$ and that the f_i are morphisms of $\mathcal{C}$. But in fact such an object and such morphisms need not exist. In the general case one must rather employ a definition of the projective limit as a certain functor from the category $\mathcal{C}$ to the category of sets, and then the question arises whether this functor is 'representable' through an object $\mathcal{L}$ and morphisms f_i as described above. In this language the crucial issue is the existence of a representing object $\mathcal{L}$ (see e.g. [9, 10, 11]).

Now one and the same projective system can frequently be regarded as part of various different categories. For instance, when describing the projective system of our interest one can restrict oneself to the category $\mathcal{F}us(\mathfrak{g})$. As we will see, when doing so a projective limit of the projective system does not exist. But one can also consider it, say, in the category of commutative rings, or in the still bigger category of vector spaces, or even in the category of sets. The existence and the precise form of the projective limit usually depend on the choice of category. In our case, however, the category $\mathcal{C} = \mathcal{F}us(\mathfrak{g})$ we start with is small, i.e. its objects

are sets, and as a consequence there exists a natural construction by which the object $\mathcal{L}$ and the morphisms f_i can be obtained in a concrete manner (in particular, $\mathcal{L}$ is again a set). Moreover, it turns out that the projective limit we obtain in the category of sets is exactly the same as the limit that we obtain in the category of commutative rings or vector spaces, which also indicates that this way of performing the limit is a quite natural.

This construction proceeds as follows. Given a projective system of objects ${}^{(h)}\mathcal{R}$ and morphisms $f_{h',h}$ of a small category $\mathcal{C}$, one regards the objects ${}^{(h)}\mathcal{R}\in\mathcal{C}$ as sets and considers the infinite direct product $\prod_{h\in I}{}^{(h)}\mathcal{R}$ of all objects of $\mathcal{C}$. The elements of this set are those maps

$$\psi:\quad I\;\to\;\dot{\bigcup_{h\in I}}\,{}^{(h)}\mathcal{R} \tag{3.4}$$

from the index set I to the disjoint union of all objects ${}^{(h)}\mathcal{R}$ which obey $\psi(h)\in{}^{(h)}\mathcal{R}$ for all $h\in I$; they are sometimes called 'generalized sequences' (ordinary sequences can be formulated in this language by considering the index set $\mathbb{N}$ with the standard ordering $\le$). The subset ${}^{(\infty)}\mathcal{R}\subset\prod_{h\in I}{}^{(h)}\mathcal{R}$ consisting of *coherent sequences*, i.e. of those generalized sequences for which

$$f_{h',h}\circ\psi(h')=\psi(h) \tag{3.5}$$

for all $h,h'\in I$ with $h\preceq h'$, is isomorphic to the projective limit. More precisely, ${}^{(\infty)}\mathcal{R}$ is isomorphic to $\mathcal{L}$ as a set, and the morphisms f_h are the projections to the components, i.e.

$$f_h(\psi):=\psi(h)\,. \tag{3.6}$$

For the projective system introduced in subsection 2.3 where $\mathcal{C}$ is the (small) category $\mathcal{F}us(\mathfrak{g})$, the projective limit is clearly *not* contained in the original category, because no object ${}^{(i)}\mathcal{R}$ of $\mathcal{F}us(\mathfrak{g})$ can possess morphisms to *all* objects ${}^{(j)}\mathcal{R}$. In order to identify nevertheless a projective limit associated to the projective system defined by (2.13), it is therefore necessary to consider the set ${}^{(\infty)}\mathcal{R}$ of coherent sequences. In accordance with the remarks above, for definiteness from now on we will simply refer to ${}^{(\infty)}\mathcal{R}$ as 'the' projective limit of the system (2.11) of WZW fusion rings.

3.2 Properties of ${}^{(\infty)}\mathcal{R}$

Let us list a few simple properties of the projective limit ${}^{(\infty)}\mathcal{R}$. First, ${}^{(\infty)}\mathcal{R}$ is a ring over $\mathbb{Z}$. The product $\psi_1\star\psi_2$ in ${}^{(\infty)}\mathcal{R}$ is defined pointwise, i.e. by the requirement that

$$(\psi_1\star\psi_2)(h):=\psi_1(h)\star\psi_2(h) \tag{3.7}$$

for all $h\in I$. This definition makes sense, i.e. for all $\psi_1,\psi_2\in{}^{(\infty)}\mathcal{R}$ also their product is in ${}^{(\infty)}\mathcal{R}$, because

$$\begin{aligned} f_{h',h}\circ(\psi_1\star\psi_2)(h') &= (f_{h',h}\circ\psi_1(h'))\star(f_{h',h}\circ\psi_2(h')) \\ &= \psi_1(h)\star\psi_2(h)=(\psi_1\star\psi_2)(h)\,; \end{aligned} \tag{3.8}$$

here in the first line the morphism property of the maps $f_{h',h}$ is used. From the definition (3.7) it is clear that the product of ${}^{(\infty)}\mathcal{R}$ is commutative and associative, and that ${}^{(\infty)}\mathcal{R}$ is unital, with the unit element being the element $\psi_\circ\in{}^{(\infty)}\mathcal{R}$ that satisfies

$$\psi_\circ(h)={}^{(h)}\varphi_\rho \tag{3.9}$$

for all $h \in I$.

Second, a conjugation $\psi \mapsto \psi^+$ can be defined on ${}^{(\infty)}\mathcal{R}$ by setting

$$\psi^+(h) := (\psi(h))^+ \tag{3.10}$$

for all $h \in I$. The conjugation ${}^{(h)}\varphi \mapsto ({}^{(h)}\varphi)^+$ on the rings ${}^{(h)}\mathcal{R}$ commutes with the projections $f_{h',h}$. As a consequence, indeed $\psi^+ \in {}^{(\infty)}\mathcal{R}$ whenever $\psi \in {}^{(\infty)}\mathcal{R}$, conjugation is an involutive automorphism of ${}^{(\infty)}\mathcal{R}$, and the unit element $\psi_\circ$ is self-conjugate.

In section 4 we will construct a countable basis ${}^{(\infty)}\mathcal{B}$ of the ring ${}^{(\infty)}\mathcal{R}$; this basis contains in particular the unit element $\psi_\circ$. For any $\psi \in {}^{(\infty)}\mathcal{B}$ and any $h \in I$, $\psi(h)$ is either zero or, up to possibly a sign, an element of the basis ${}^{(h)}\mathcal{B}$ of ${}^{(h)}\mathcal{R}$. Also, while by construction the structure constants in the basis ${}^{(\infty)}\mathcal{B}$ are integers, there seems to be no reason why they should be non-negative. Accordingly, an interpretation of the limit ${}^{(\infty)}\mathcal{R}$ as the representation ring of some underlying algebraic structure is even less obvious than in the case of the fusion rings ${}^{(h)}\mathcal{R}$.[2] In particular, in section 5 we will see that ${}^{(\infty)}\mathcal{R}$ does not coincide with the representation ring $\overline{\mathcal{R}}$ of the simple Lie algebra $\bar{\mathfrak{g}} \subset \mathfrak{g}$, but rather that it contains $\overline{\mathcal{R}}$ as a tiny proper subring.

As it turns out, the fusion product of two elements of ${}^{(\infty)}\mathcal{B}$ is generically *not* a finite linear combination of elements of ${}^{(\infty)}\mathcal{B}$, or in other words, ${}^{(\infty)}\mathcal{B}$ does not constitute an ordinary basis of ${}^{(\infty)}\mathcal{R}$. Rather, it must be regarded as a topological basis. For this interpretation to make sense, a suitable topology on ${}^{(\infty)}\mathcal{R}$ must be defined. This will be achieved in the next subsection.

3.3 The limit topology of ${}^{(\infty)}\mathcal{R}$

The fusion rings ${}^{(h)}\mathcal{R}$ can be considered as topological spaces by simply endowing them with the discrete topology, i.e. by declaring every subset to be open (and hence also every subset to be closed). The projective limit ${}^{(\infty)}\mathcal{R}$ then becomes a topological space in a natural manner, namely by defining its topology as the coarsest topology in which all projections $f_h\colon\ {}^{(\infty)}\mathcal{R} \to {}^{(h)}\mathcal{R}$ are continuous; this will be called the *limit topology* on ${}^{(\infty)}\mathcal{R}$.

More explicitly, the limit topology on ${}^{(\infty)}\mathcal{R}$ is described by the property that any open set in ${}^{(\infty)}\mathcal{R}$ is an (arbitrary, i.e. not necessarily finite nor even countable) union of elements of

$$\Omega := \{ f_h^{-1}(M) \mid h \in I,\ M \subseteq {}^{(h)}\mathcal{R} \}\,, \tag{3.11}$$

i.e. of the set of all pre-images of all sets in any of the fusion rings ${}^{(h)}\mathcal{R}$.
Note that we need not require to take also finite intersections of these pre-images. This is because Ω is closed under taking finite intersections, as can be seen as follows. Let $\omega_i \in \Omega$ for $i = 1, 2, \ldots, N$; by definition, each of the ω_i can be written as $\omega_i = f_{h_i}^{-1}(M_i)$ for some heights $h_i \in I$ and some subsets $M_i \subseteq {}^{(h_i)}\mathcal{R}$. Denote then by h the smallest common multiple of the h_i for $i = 1, 2, \ldots, N$. Because of (3.1) we have $f_{h_i} = f_{h,h_i} \circ f_h$, so that

$$f_{h_i}^{-1}(M_i) = f_h^{-1}(f_{h,h_i}^{-1}(M_i)) = f_h^{-1}(\tilde{M}_i)\,, \tag{3.12}$$

where for all $i = 1, 2, \ldots, N$ the sets $\tilde{M}_i := f_{h,h_i}^{-1}(M_i)$ are subsets of ${}^{(h)}\mathcal{R}$. Because of $\bigcap_{i=1}^N \tilde{M}_i \subseteq {}^{(h)}\mathcal{R}$, it thus follows that

$$\bigcap_{i=1}^N \omega_i = \bigcap_{i=1}^N f_h^{-1}(\tilde{M}_i) = f_h^{-1}[\bigcap_{i=1}^N \tilde{M}_i] \tag{3.13}$$

[2] The latter can e.g. be regarded as the representation rings of the 'quantum symmetry' of the associated WZW theories. However, so far there is no agreement on the precise nature of those quantum symmetries.

is an element of the set Ω (3.11). Thus Ω is closed under taking finite intersections, as claimed. As a consequence of this property of Ω, in particular any non-empty open set in ${}^{(\infty)}\!\mathcal{R}$ contains a subset which is of the form $f_h^{-1}(M)$ for some $h\in I$ and some $M \subseteq {}^{(h)}\!\mathcal{R}$; for later reference, we call this fact the 'pre-image property' of the non-empty open sets in ${}^{(\infty)}\!\mathcal{R}$.

Note that the limit topology on ${}^{(\infty)}\!\mathcal{R}$ is *not* the discrete one, but finer. To see this, suppose the limit topology were the discrete one. Then for any $\psi\in{}^{(\infty)}\!\mathcal{R}$ the one-element set $\{\psi\}$ would be open and hence a union of sets in Ω (3.11); but as $\{\psi\}$ just contains one single element, this means that it even has to belong itself to Ω. This in turn means that there would exist $h\in I$ and $M \subseteq {}^{(h)}\!\mathcal{R}$ such that $\{\psi\} = f_h^{-1}(M)$, and hence simply $M = \{\psi(h)\}$. This, however, would imply that each element $\psi \in {}^{(\infty)}\!\mathcal{R}$ would already be determined uniquely by the value $\psi(h)$ for a single height h. From the explicit description of ${}^{(\infty)}\!\mathcal{R}$ as a space of coherent generalized sequences, it follows that this is definitely not true. Thus the assumption that the limit topology is the discrete one leads to a contradiction.

Whenever two elements $\psi, \psi' \in {}^{(\infty)}\!\mathcal{R}$ are distinct, there exists some height $h\in I$ such that $\psi(h) \neq \psi'(h)$. The open subsets $\omega := f_h^{-1}(\{\psi(h)\})$ and $\omega' := f_h^{-1}(\{\psi'(h)\})$ then satisfy $\psi\in\omega$ and $\psi'\in\omega'$ as well as $\omega\cap\omega' = \emptyset$. This means that when endowed with the limit topology, ${}^{(\infty)}\!\mathcal{R}$ is a Hausdorff space.

4 A distinguished basis ${}^{(\infty)}\!\mathcal{B}$ of ${}^{(\infty)}\!\mathcal{R}$

In this section we construct a (topological) basis ${}^{(\infty)}\!\mathcal{B}$ of the projective limit ${}^{(\infty)}\!\mathcal{R}$ of WZW fusion rings.

4.1 A linearly independent subset of ${}^{(\infty)}\!\mathcal{R}$

We start by defining the subset ${}^{(\infty)}\!\mathcal{B} \subset {}^{(\infty)}\!\mathcal{R}$ as the set of all those elements $\psi\in{}^{(\infty)}\!\mathcal{R}$ which for every $h\in I$ satisfy

$$\psi(h) = \epsilon_h \cdot {}^{(h)}\!\varphi_a \tag{4.1}$$

for some

$$a\in{}^{(h)}\!P \qquad \text{and} \qquad \epsilon_h\in\{0,\pm1\} \tag{4.2}$$

(i.e. for each height h the fusion ring element $\psi(h) \in {}^{(h)}\!\mathcal{R}$ is either zero or, up to a sign, an element of the distinguished basis ${}^{(h)}\!\mathcal{B}$), and for which in addition not all of the prefactors ϵ_h vanish and $\epsilon_h = 1$ for the smallest $h\in I$ for which $\epsilon_h \neq 0$. The latter requirement ensures that $-\psi \notin {}^{(\infty)}\!\mathcal{B}$ for all $\psi\in{}^{(\infty)}\!\mathcal{B}$.

Note that at this point we cannot tell yet whether the set ${}^{(\infty)}\!\mathcal{B}$ is large enough to generate the whole ring ${}^{(\infty)}\!\mathcal{B}$; in fact, it is even not yet clear whether ${}^{(\infty)}\!\mathcal{B}$ is non-empty. These issues will be dealt with in subsections 4.2 to 4.4 below, where we will in particular see that the set ${}^{(\infty)}\!\mathcal{B}$ is countably infinite. However, what we already can see is that the set ${}^{(\infty)}\!\mathcal{B}$ is linearly independent. To prove this, consider any set of finitely many distinct elements ψ_i, $i = 1, 2, ... , N$ of ${}^{(\infty)}\!\mathcal{B}$. We first show that to any pair $i, j \in \{1, 2, ... , N\}$ there exists a height $h_{ij}\in I$ such that

(i) $\psi_i(h_{ij}) \neq 0$ and $\psi_j(h_{ij}) \neq 0$ and

(ii) $\psi_i(h_{ij}) \neq \pm\psi_j(h_{ij})$.

To see this, assume that the statement is wrong, i.e. that for each height h either one of the

elements $\psi_i(h)$ and $\psi_j(h)$ of ${}^{(h)}\mathcal{R}$ vanishes, or one has $\psi_i(h) = \pm\psi_j(h)$. Now because of $\psi_i \neq 0$ and $\psi_j \neq 0$ there exists heights h_i and h_j with $\psi_i(h_i) \neq 0$ and $\psi_j(h_j) \neq 0$. This implies that also $\psi_i(\tilde{h}_{ij}) \neq 0$ and $\psi_j(\tilde{h}_{ij}) \neq 0$ for $\tilde{h}_{ij} := h_i h_j$. By our assumption it then follows that $\psi_j(\tilde{h}_{ij}) = \pm\psi_i(\tilde{h}_{ij})$, which in turn implies that $\psi_j(h_i) = \pm\psi_i(h_i) \neq 0$. Now this conclusion actually extends to arbitrary heights h. Namely, from the previous result we know that for any h the elements $\psi_i(h\tilde{h}_{ij})$ and $\psi_j(h\tilde{h}_{ij})$ must both be non-zero. By our assumption this implies that $\psi_j(h\tilde{h}_{ij}) = \pm\psi_i(h\tilde{h}_{ij})$. Projecting this equation down to the height h, it follows that $\psi_j(h) = \pm\psi_i(h)$. Since h was arbitrary, it follows that in fact $\psi_j = \pm\psi_i$, and hence (as $-\psi_i$ is not in ${}^{(\infty)}\mathcal{B}$) that $\psi_j = \psi_i$. This is in contradiction to the requirement that all ψ_i should be distinct. Thus our assumption must be wrong, which proves that (i) and (ii) are fulfilled.

Applying now the properties (i) and (ii) for any pair $i, j \in \{1, 2, \dots, N\}$ with $i \neq j$, it follows that at the height $h := \prod_{i,j;i<j} h_{ij}$ we have

(i) $\psi_i(h) \neq 0$ for all $i = 1, 2, \dots, N$ and

(ii) $\psi_i(h) \neq \pm\psi_j(h)$ for all $i, j \in \{1, 2, \dots, N\}$, $i \neq j$.

Thus all the elements $\psi_i(h)$ of ${}^{(h)}\mathcal{R}$ are distinct and, up to sign, elements of the distinguished basis ${}^{(h)}\mathcal{B}$. This implies in particular that the only solution of the equation $\sum_{i=1}^{N} \xi_i \psi_i(h) = 0$ is $\xi_i = 0$ for $i = 1, 2, \dots, N$, which in turn shows that also the equation $\sum_{i=1}^{N} \xi_i \psi_i = 0$ has only this solution. Thus, as claimed, the ψ_i are linearly independent elements of ${}^{(\infty)}\mathcal{R}$.

4.2 ${}^{(\infty)}\mathcal{B}$ generates all of ${}^{(\infty)}\mathcal{R}$

Next we claim that the set ${}^{(\infty)}\mathcal{B}$ spans ${}^{(\infty)}\mathcal{R}$ in the sense that the closure (in the limit topology) of the linear span of ${}^{(\infty)}\mathcal{B}$ in ${}^{(\infty)}\mathcal{R}$, i.e. of the set

$$\langle {}^{(\infty)}\mathcal{B} \rangle \equiv \mathrm{span}_{\mathbb{Z}}({}^{(\infty)}\mathcal{B}) \tag{4.3}$$

of finite $\mathbb{Z}$-linear combinations of elements of ${}^{(\infty)}\mathcal{B}$, is already all of ${}^{(\infty)}\mathcal{R}$.

To prove this, assume that the statement is wrong, or in other words, that the set

$$\mathcal{S} := {}^{(\infty)}\mathcal{R} \setminus \overline{\langle {}^{(\infty)}\mathcal{B} \rangle} \tag{4.4}$$

is non-empty. By definition, the set $\mathcal{S}$ is open, and hence because of the pre-image property it contains a subset $\mathcal{M} \subseteq \mathcal{S}$ of the form $\mathcal{M} = f_h^{-1}(M)$ for some $h \in I$ and some $M \subseteq {}^{(h)}\mathcal{R}$. Further, as an immediate consequence of the construction that we will present in the subsections 4.3 and 4.4, for each $a \in {}^{(h)}P$ there exists an element (in fact, infinitely many elements) $\psi_a \in {}^{(\infty)}\mathcal{B}$ such that $f_h(\psi_a) = {}^{(h)}\varphi_a$ (namely, we need to prescribe the value of $\psi_a(p)$ only for the finitely many prime factors p of h). Now choose some $y \in M$, decompose it with respect to the basis ${}^{(h)}\mathcal{B}$ of ${}^{(h)}\mathcal{R}$, i.e. $y = \sum_{a \in {}^{(h)}P} n_a \, {}^{(h)}\varphi_a$ with $n_a \in \mathbb{Z}$, and define $\eta := \sum_{a \in {}^{(h)}P} n_a \, \psi_a$. Then, on one hand, by construction we have $f_h(\eta) = y$, i.e. $\eta \in \mathcal{M}$, and hence $\eta \in \mathcal{S}$, while on the other hand η is a finite linear combination of elements of ${}^{(\infty)}\mathcal{B}$ (since y is a finite linear combination of elements of ${}^{(h)}\mathcal{B}$), and hence $\eta \in \langle {}^{(\infty)}\mathcal{B} \rangle \subseteq \overline{\langle {}^{(\infty)}\mathcal{B} \rangle}$. By the definition (4.4) of $\mathcal{S}$, this is a contradiction, and hence our assumption must be wrong.

Together with the result of the previous subsection we thus see that ${}^{(\infty)}\mathcal{B}$ is a (topological) basis of ${}^{(\infty)}\mathcal{R}$.

4.3 Distinguished sequences of integral weights

We will now construct all elements of the projective limit ${}^{(\infty)}\!\mathcal{R}$ which belong to the subset ${}^{(\infty)}\!\mathcal{B}$ as introduced in subsection 4.1. These are obtained as generalized sequences ψ satisfying both (3.5) and the defining relation (4.1) of ${}^{(\infty)}\!\mathcal{B}$. More specifically, we construct sequences $(a_h)_{h\in I}$ of labels $a_h \in {}^{(h)}\!P$ and associated signs $\eta(a_h)$ such that all those ψ which are of the form

$$\psi(h) = \eta(a_h)\, {}^{(h)}\!\varphi_{a_h} \tag{4.5}$$

belong to the subset ${}^{(\infty)}\!\mathcal{B} \subset {}^{(\infty)}\!\mathcal{R}$. When applied to (4.5), the requirement (3.5) amounts to

$$f_{\ell h,h}({}^{(\ell h)}\!\varphi_{a_{\ell h}}) = \eta(a_{\ell h})\eta(a_h) \cdot {}^{(h)}\!\varphi_{a_h}\,, \tag{4.6}$$

which in view of the definition (2.13) of $f_{\ell h,h}$ is equivalent to

$$a_{\ell h} = w(a_h) \qquad \text{for some } w \in {}^{(h)}\!W \tag{4.7}$$

and

$$\eta(a_{\ell h})\eta(a_h) = \epsilon_\ell(a_{\ell h})\,. \tag{4.8}$$

To start the construction of the elements of ${}^{(\infty)}\!\mathcal{B}$, we first concentrate our attention to integral weights of the height h theory which are not necessarily integrable and which are considered as defined only modulo $h\overline{L}^\vee$; we denote these weights by b_h. Suppose then that we prescribe for each prime p such a weight b_p and that these weights satisfy in addition the restriction that for any two primes p, p' they differ by an element of the coroot lattice,

$$b_p - b_{p'} \in \overline{L}^\vee\,. \tag{4.9}$$

We claim that there then exists a sequence $(b_h)_{h\in I}$ which for prime heights takes the prescribed values b_p and for which the relation

$$b_{h'} = b_h \bmod h\overline{L}^\vee \tag{4.10}$$

holds for all h, h' with $h \preceq h'$.

To prove this assertion, we display such a sequence explicitly. To this end, let

$$h =: \prod_{\substack{j\\ p_j|h}} p_j^{n_j} \tag{4.11}$$

denote the decomposition of h into prime factors, and define

$$h_i := \frac{h}{p_i^{n_i}} \tag{4.12}$$

and

$$[h_i]^{-1}_{p_i^{n_i}} := (h_i)^{-1} \bmod p_i^{n_i}\,. \tag{4.13}$$

Then we set

$$b_h := b_{p_1} + \sum_{\substack{i\neq 1\\ p_i|h}} h_i\, [h_i]^{-1}_{p_i^{n_i}}\, (b_{p_i} - b_{p_1})\,. \tag{4.14}$$

Recall that b_h is defined only modulo $h\overline{L}^\vee$. In (4.14) p_1 is any of the prime divisors of h; it has been singled out only in order to make the formula for b_h to look as simple as possible, and in fact b_h does not depend (modulo $h\overline{L}^\vee$) on the choice of p_1. To see this, let $b_h^{(2)}$ denote the number obtained analogously as in (4.14), but with p_1 replaced by some other prime factor p_2 of h. Then

$$\begin{aligned} b_h - b_h^{(2)} = b_{p_1} - b_{p_2} + \sum_{\substack{i\neq 1,2 \\ p_i|h}} h_i \left[h_i\right]^{-1}_{p_i^{n_i}} (b_{p_2} - b_{p_1}) \\ + h_2 \left[h_2\right]^{-1}_{p_2^{n_2}} (b_{p_2} - b_{p_1}) - h_1 \left[h_1\right]^{-1}_{p_1^{n_1}} (b_{p_1} - b_{p_2}) \,. \end{aligned} \tag{4.15}$$

Using the fact that h_i is divisible by $p_j^{n_j}$ for all primes p_j dividing h except for $j=i$, and that $h_i\big[h_i\big]^{-1}_{p_i^{n_i}} = 1 \bmod p_i^{n_i}$, it is easily checked that the right hand side of this expression vanishes modulo $p_j^{n_j}\overline{L}^\vee$ for all p_j dividing h, and hence, using (4.9), also vanishes modulo $h\overline{L}^\vee$.

To establish the coherence property (4.10), we now consider two heights h, h' such that $h|h'$. Then we set

$$h' =: \prod_{\substack{j \\ p_j|h'}} p_j^{n'_j} \tag{4.16}$$

and $h'_i := h'/p_i^{n'_i}$, and without loss of generality we can assume that p_1 divides h as well as h'. By the definition (4.14) we then have

$$b_{h'} - b_h = \sum_{\substack{i\neq 1 \\ p_i|h}} \Big\{ h'_i \left[h'_i\right]^{-1}_{p_i^{n'_i}} - h_i \left[h_i\right]^{-1}_{p_i^{n_i}} \Big\} (b_{p_i} - b_{p_1}) + \sum_{\substack{i \\ p_i|h',\, p_i\not| h}} h'_i \left[h'_i\right]^{-1}_{p_i^{n'_i}} (b_{p_i} - b_{p_1}) \,; \tag{4.17}$$

again it is straightforward to verify that this vanishes modulo $p_j^{n_j}\overline{L}^\vee$ for all p_j dividing h. This shows that the property (4.10) is satisfied for the sequence defined by (4.14) as claimed.

Next we note that we did not require that the prescribed values b_p lie on the Weyl orbit of an integrable weight at height p, but rather they may also lie on the boundary of some Weyl chamber of ${}^{(p)}W$. However, if b_p does belong to the Weyl orbit of an integrable weight, then also each weight b_h with $p|h$ is on the Weyl orbit of an integrable weight at height h. Namely, because of the property (4.10) we have in particular

$$b_h = b_p + p^n\,\beta \tag{4.18}$$

for some $\beta \in \overline{L}^\vee$. Hence, assuming that b_h is left invariant by some $w \in {}^{(h)}W$, i.e. that $b_h = w(b_h) \equiv \overline{w}(b_h) + h\gamma$ for some element γ of the coroot lattice, it follows that

$$\begin{aligned} \overline{w}(b_p) &= \overline{w}(b_h - p^n\beta) = \overline{w}(b_h) - p^n\,\overline{w}(\beta) \\ &= b_h - h\gamma - p^n\,\overline{w}(\beta) = b_p + p^n\,(\beta - \overline{w}(\beta)) - h\gamma \,. \end{aligned} \tag{4.19}$$

Since by assumption the only element of ${}^{(p)}W$ which leaves the weight b_p invariant is the identity, it follows that $\overline{w} = \mathsf{id}$ and $\gamma = 0$, implying that also the only element of ${}^{(h)}W$ that leaves b_h invariant is the identity, which is equivalent to the claimed property.

Our next task is to investigate to what extent the sequence $(b_h)_{h\in I}$ is characterized by the prescribed values b_p at prime heights and by the requirement (4.10). To this end let $(\tilde b_h)_{h\in I}$ be

another such sequence, i.e. a sequence such that $\tilde b_p = b_p$ for all primes p and $\tilde b_{h'} - \tilde b_h \in h\overline{L}^\vee$ for $h|h'$. First we observe that for h and h' coprime, the properties

$$\tilde b_{hh'} = \tilde b_h \bmod h\overline{L}^\vee \qquad \text{and} \qquad \tilde b_{hh'} = \tilde b_{h'} \bmod h'\overline{L}^\vee \tag{4.20}$$

fix $\tilde b_{hh'}$ already uniquely (modulo $hh'\overline{L}^\vee$), so that the whole freedom is parametrized by the freedom in the choice of $\tilde b_h$ at heights which are a prime power. Concerning the latter freedom, we claim that for any prime p there is a sequence of elements $\beta_p^{(j)}$ of the coroot lattice $\overline{L}^\vee$ which are defined modulo $p\overline{L}^\vee$ such that the most general choice of $\tilde b_{p^n}$ reads

$$\tilde b_{p^n} = b_{p^n} + \sum_{j=1}^{n-1} \beta_p^{(j)}\, p^j \tag{4.21}$$

with b_{p^n} defined according to (4.14), i.e. simply $b_{p^n} = b_p$. This statement is proven by induction. For $n = 1$ it is trivially fulfilled. Further, assuming that (4.21) is satisfied for some $n \geq 1$ and setting $\gamma := \tilde b_{p^{n+1}} - b_{p^{n+1}}$ (defined modulo $p^{n+1}\overline{L}^\vee$), one has

$$\tilde b_{p^{n+1}} - \tilde b_{p^n} = \gamma - \sum_{j=1}^{n-1} \beta_p^{(j)}\, p^j\,. \tag{4.22}$$

By the required properties of the sequence $(\tilde b_h)$, the left hand side of this formula must vanish modulo $p^n\overline{L}^\vee$, and hence we have

$$\gamma = \beta_p^{(n)}\, p^n + \sum_{j=1}^{n-1} \beta_p^{(j)}\, p^j = \sum_{j=1}^{n} \beta_p^{(j)}\, p^j \tag{4.23}$$

for some $\beta_p^{(n)} \in \overline{L}^\vee$. This shows that $\tilde b_{p^{n+1}}$ is again of the form described by (4.21); furthermore, as γ is defined modulo $p^{n+1}\overline{L}^\vee$, $\beta_p^{(n)}$ is defined modulo $p\overline{L}^\vee$ as required, and hence the proof of the formula (4.21) is completed.

With these results we are now in a position to give a rather explicit description of the allowed sequences $\tilde b_h$. Namely, we can parametrize the general form of $\tilde b_h$ in terms of the freedom in $\tilde b_{p^n}$ according to

$$\begin{aligned} \tilde b_h &= \tilde b_{p_1^{n_1}} + \sum_{\substack{i\neq 1\\ p_i|h}} h_i\, [h_i]^{-1}_{p_i^{n_i}}\, (\tilde b_{p_i^{n_i}} - \tilde b_{p_1^{n_1}}) \\ &= b_{p_1} + \sum_{j=1}^{n_1-1} \beta_{p_1}^{(j)}\, p_1^j + \sum_{\substack{i\neq 1\\ p_i|h}} h_i\, [h_i]^{-1}_{p_i^{n_i}}\, (b_{p_i} - b_{p_1}) + \sum_{\substack{i\neq 1\\ p_i|h}} h_i\, [h_i]^{-1}_{p_i^{n_i}}\, [\sum_{j=1}^{n_i-1} \beta_{p_i}^{(j)}\, p_i^j - \sum_{j=1}^{n_1-1} \beta_{p_1}^{(j)}\, p_1^j] \\ &= b_h + \sum_{\substack{i\neq 1\\ p_i|h}} h_i\, [h_i]^{-1}_{p_i^{n_i}}\, [\sum_{j=1}^{n_i-1} \beta_{p_i}^{(j)}\, p_i^j - \sum_{j=1}^{n_1-1} \beta_{p_1}^{(j)}\, p_1^j] + \sum_{j=1}^{n_1-1} \beta_{p_1}^{(j)}\, p_1^j\,. \end{aligned} \tag{4.24}$$

Further, any such sequence fulfills the consistency requirement that $\tilde{b}_{h'} - \tilde{b}_h \in h\overline{L}^\vee$ for heights h, h' with $h|h'$. Namely, in this case the formula (4.24) yields

$$\begin{aligned} \tilde{b}_{h'} - \tilde{b}_h = & \sum_{\substack{i\neq 1\\ p_i|h}} \Big\{ h'_i\, [h'_i]^{-1}_{p_i^{n'_i}} \,[\sum_{j=1}^{n'_i-1} \beta^{(j)}_{p_i}\, p_i^j - \sum_{j=1}^{n'_1-1} \beta^{(j)}_{p_1}\, p_1^j] - h_i\, [h_i]^{-1}_{p_i^{n_i}} \,[\sum_{j=1}^{n_i-1} \beta^{(j)}_{p_i}\, p_i^j - \sum_{j=1}^{n_1-1} \beta^{(j)}_{p_1}\, p_1^j] \Big\} \\ & + \sum_{\substack{i\\ p_i|h',\, p_i\not| h}} h'_i\, [h'_i]^{-1}_{p_i^{n'_i}} \,[\sum_{j=1}^{n'_i-1} \beta^{(j)}_{p_i}\, p_i^j - \sum_{j=1}^{n'_1-1} \beta^{(j)}_{p_1}\, p_1^j] + \sum_{j=n_1}^{n'_1-1} \beta^{(j)}_{p_1}\, p_1^j \mod h\overline{L}^\vee\,. \end{aligned} \tag{4.25}$$

Once more one can easily check that this expression vanishes modulo $p_j^{n_j}\overline{L}^\vee$ for all primes p_j that divide h, and hence vanishes modulo $h\overline{L}^\vee$. Thus the consistency requirement indeed is satisfied.

4.4 Distinguished sequences of integrable weights and the basis ${}^{(\infty)}\mathcal{B}$

What we have achieved so far is a characterization of all sequences $(b_h)_{h\in I}$ of integral weights defined modulo $h\overline{L}^\vee$ that satisfy (4.10). We now use this result to construct sequences $(a_h)_{h\in I}$ of highest weights which satisfy the requirement (4.7) and of which infinitely many are *integrable* weights, with all non-integrable weights being equal to zero. We start by prescribing integrable weights $a_p \in {}^{(p)}P$ for all primes p with $p \geq g^\vee$, and set $b_p = a_p$ for $p \geq g^\vee$, while for all primes $p < g^\vee$ we choose arbitrary weights b_p (which are necessarily non-integrable). Next we employ the previous results to find the sequences $(b_h)_{h\in I}$. Finally we define a_h for any arbitrary height h as follows. If b_h lies on the boundary of a Weyl chamber with respect to ${}^{(h)}W$, then we set $a_h = 0$. Otherwise there are a unique element $\overline{w}_h \in \overline{W}$ and a unique [3] element $\gamma_h \in \overline{L}^\vee$ such that $\overline{w}_h(b_h) + h\gamma_h$ is integrable, and in this case we set

$$a_h := \overline{w}_h(b_h) + h\gamma_h\,. \tag{4.26}$$

By construction, the weights a_h have the following properties. If $a_{h'} = 0$ for some height h', then $b_{h'}$ is on the boundary of a Weyl chamber with respect to ${}^{(h')}W$; for any h dividing h', it then follows from $b_{h'} - b_h \in h\overline{L}^\vee$ that b_h is on the boundary of a Weyl chamber with respect to ${}^{(h)}W$, and hence we also have $a_h = 0$. On the other hand, if $b_{h'}$ is equivalent with respect to ${}^{(h')}W$ to an integrable weight $a_{h'} \in {}^{(h')}P$, then it is a fortiori equivalent to $a_{h'}$ with respect to the larger group ${}^{(h)}W$, and then the property $b_{h'} - b_h \in h\overline{L}^\vee$ implies that also b_h is equivalent with respect to ${}^{(h)}W$ to $a_{h'}$, and hence that the associated weight a_h is integrable at height h and is equivalent with respect to ${}^{(h)}W$ to $a_{h'}$, too. Thus a_h and $a_{h'}$ are on the same orbit with respect to ${}^{(h)}W$ whenever h divides h', and hence (4.7) holds as promised.

Note that by construction for all $\bar{\mathfrak{g}}$ except $\bar{\mathfrak{g}} = A_1$ the sequences so obtained contain some zero weights. However, any sequence which contains at least one non-zero weight contains in fact infinitely many non-zero (and hence integrable) weights.

The final step is now to define

$$\psi(h) := \eta(a_h)\, {}^{(h)}\varphi_{a_h} \tag{4.27}$$

[3] To be precise, because the weights a_h are defined only modulo $h\overline{L}^\vee$, γ_h is only unique once a definite representative of the equivalence class of weights that is described by a_h is chosen.

as in (4.5), where a_h is as constructed above, and where

$$\eta(a_h) := \begin{cases} \mathrm{sign}(\overline{w}_h) & \text{for } a_h \in {}^{(h)}W\,, \\ 0 & \text{for } a_h = 0\,. \end{cases} \tag{4.28}$$

To show that ψ is an element of the projective limit, it only remains to check the property (4.8) of the prefactor $\eta(a_h)$. For $a_h = 0$ (4.8) just reads $0 = 0$ and is trivially satisfied. For $a_h \in {}^{(h)}P$, the previous results show that $a_{\ell h} = w_{\ell h} \circ w_0 \circ w_h^{-1}(a_h)$, where w_0 is the Weyl translation relating b_h and $b_{\ell h}$, so that

$$\epsilon_\ell(a_{\ell h}) = \mathrm{sign}(w_{\ell h} \circ w_0 \circ w_h) = \mathrm{sign}(\overline{w}_{\ell h}) \cdot \mathrm{sign}(\overline{w}_h)\,. \tag{4.29}$$

In view of the definition (4.28) of $\eta(a_h)$, this is precisely the required relation (4.8). We conclude that the basis ${}^{(\infty)}\mathcal{B}$ of ${}^{(\infty)}\mathcal{R}$ precisely consists of the elements (4.27). In particular, ${}^{(\infty)}\mathcal{B}$ is countably infinite.

5 The fusion ring of $\bar{\mathfrak{g}}$

As already pointed out in the introduction, it is expected that in the limit of infinite level of WZW theories somehow the simple Lie algebra $\bar{\mathfrak{g}}$ which is the horizontal subalgebra of $\mathfrak{g}$ and its representation theory should play a rôle. More specifically, one might think that the representation ring $\overline{\mathcal{R}}$ of $\bar{\mathfrak{g}}$ emerges. As we will demonstrate below, indeed this ring shows up, but it is only a proper subring of the projective limit ${}^{(\infty)}\mathcal{R}$ we constructed, and almost all elements of ${}^{(\infty)}\mathcal{R}$ are *not* contained in the ring $\overline{\mathcal{R}}$.

Let us describe $\overline{\mathcal{R}}$ and its connection with the category $\mathcal{F}us(\mathfrak{g})$ in some detail. $\overline{\mathcal{R}}$ is defined as the ring over $\mathbb{Z}$ of all isomorphism classes of finite-dimensional $\bar{\mathfrak{g}}$-representations, with the ring product the ordinary tensor product of $\bar{\mathfrak{g}}$-representations (or, equivalently, the pointwise product of the characters of these representations). This ring $\overline{\mathcal{R}}$ is a fusion ring with an infinite basis. The elements $\bar{\varphi}_a$ of a distinguished basis of $\overline{\mathcal{R}}$ are labelled by the (shifted) highest weights of irreducible finite-dimensional $\bar{\mathfrak{g}}$-representations, i.e. by elements of the set

$$\bar{P} := \{a \in \overline{L}^{\mathrm{w}} \mid 0 < a^i \text{ for } i = 1, 2, \dots, r\}\,. \tag{5.1}$$

Now for any $h \in I$ let us define the map $\bar{f}_h\colon\ \overline{\mathcal{R}} \to {}^{(h)}\mathcal{R}$ as follows. If $a \in \bar{P}$ lies on the boundary of some Weyl chamber with respect to ${}^{(h)}W$, we set $\bar{f}_h(\bar{\varphi}_a) := 0$; otherwise there exist a unique $a' \in {}^{(h)}P$ and a unique $w \in {}^{(h)}W$ such that $w(a) = a'$, and in this case we set

$$\bar{f}_h(\bar{\varphi}_a) := \epsilon(a) \cdot {}^{(h)}\varphi_{a'} \tag{5.2}$$

with $\epsilon(a) = \mathrm{sign}(w)$. As in the case of the maps $f_{\ell h,h}$ (2.13), we will consider (5.2) as covering all cases, i.e. set $\epsilon(a) = 0$ if a lies on the boundary of a Weyl chamber at height h.

To analyze the relation between the ring $\overline{\mathcal{R}}$ and the category $\mathcal{F}us(\mathfrak{g})$, we first recall the expressions

$$\overline{\mathcal{N}}_{a,b}^{\ \ c} = \sum_{\overline{w} \in \overline{W}} \mathrm{sign}(\overline{w})\, \mathrm{mult}_b(\overline{w}(c) - a) \tag{5.3}$$

for the Littlewood–Richardson coefficients (or tensor product coefficients) of $\bar{\mathfrak{g}}$ [12, 13] and

$$ {}^{(h)}\!\mathcal{N}_{a,b}{}^{c} = \sum_{w\in{}^{(h)}W} \mathrm{sign}(w)\,\mathrm{mult}_b(w(c)-a) \tag{5.4} $$

for the fusion rule coefficients, i.e. the structure constants of the WZW fusion ring ${}^{(h)}\!\mathcal{R}$ [5, 14, 15, 16]. Here $\mathrm{mult}_a(b)$ denotes the multiplicity of the (shifted) weight b in the $\bar{\mathfrak{g}}$-representation with (shifted) highest weight a. It will be convenient to extend the validity of (5.3) by adopting it as a definition of $\overline{\mathcal{N}}_{a,b}{}^{c}$ for arbitrary (i.e., not necessarily lying in $\bar{P}$) integral weights a and c, and also extend it to arbitrary integral weights b that do not lie on the boundary of any Weyl chamber with respect to $\overline{W}$ by setting

$$ \mathrm{mult}_b(c) := \mathrm{sign}(\overline{w}_b)\,\mathrm{mult}_{\overline{w}_b(b)}(c)\,, \tag{5.5} $$

with $\overline{w}_b$ the unique element of $\overline{W}$ such that $\overline{w}_b(b)\in\bar{P}$.

The multiplicities $\mathrm{mult}_a(b)$ are invariant under the Weyl group $\overline{W}$, i.e. $\mathrm{mult}_a(\overline{w}(b)) = \mathrm{mult}_a(b)$ for all $\overline{w}\in\overline{W}$. As a consequence, the numbers $\overline{\mathcal{N}}_{a,b}{}^{c}$ and ${}^{(h)}\!\mathcal{N}_{a,b}{}^{c}$ are related by [4]

$$ {}^{(h)}\!\mathcal{N}_{a,b}{}^{c} = \frac{1}{|\overline{W}|} \sum_{w\in{}^{(h)}W} \mathrm{sign}(w)\,\overline{\mathcal{N}}_{a,b}{}^{w(c)}\,. \tag{5.6} $$

The invariance of $\mathrm{mult}_a(b)$ under $\overline{W}$ also implies that for arbitrary integral weights a, b and c the symmetry property

$$ \overline{\mathcal{N}}_{a,b}{}^{c} = \overline{\mathcal{N}}_{b,a}{}^{c} \tag{5.7} $$

follows from the analogous property of the Littlewood–Richardson coefficients with $a,b,c\in\bar{P}$, and that

$$ \begin{aligned} \overline{\mathcal{N}}_{\overline{w}_1(a),b}{}^{\overline{w}_2(c)} &= \sum_{\overline{w}\in\overline{W}} \mathrm{sign}(\overline{w})\,\mathrm{mult}_b(\overline{w}\,\overline{w}_2(c)-\overline{w}_1(a)) \\ &= \sum_{\overline{w}\in\overline{W}} \mathrm{sign}(\overline{w})\,\mathrm{mult}_b(\overline{w}_1^{-1}\overline{w}\,\overline{w}_2(c)-a) = \mathrm{sign}(\overline{w}_1\overline{w}_2)\cdot\overline{\mathcal{N}}_{a,b}{}^{c}\,. \end{aligned} \tag{5.8} $$

When combined with the symmetry property (5.7), the latter formula yields

$$ \overline{\mathcal{N}}_{\overline{w}_1(a),\overline{w}_2(b)}{}^{\overline{w}_3(c)} = \mathrm{sign}(\overline{w}_1\overline{w}_2\overline{w}_3)\cdot\overline{\mathcal{N}}_{a,b}{}^{c}\,. \tag{5.9} $$

To obtain information about the effect of affine Weyl transformations on the labels of $\overline{\mathcal{N}}_{a,b}{}^{c}$, we consider an alternating sum over the Weyl group ${}^{(h)}W$. We have

$$ \begin{aligned} \sum_{w_2\in{}^{(h)}W} \mathrm{sign}(w_2)\,\overline{\mathcal{N}}_{w_1(a),b}{}^{w_2(c)} &= \sum_{\substack{\overline{w},\overline{w}_2\in\overline{W}\\ \beta_2\in\bar{L}^\vee}} \mathrm{sign}(\overline{w})\mathrm{sign}(\overline{w}_2)\,\mathrm{mult}_b(\overline{w}\,\overline{w}_2(c)+h\overline{w}(\beta_2)-\overline{w}_1(a)-h\beta_1) \\ &= \sum_{\overline{w},\overline{w}_2\in\overline{W}}\ \sum_{\beta\in\bar{L}^\vee} \mathrm{sign}(\overline{w}\,\overline{w}_2)\,\mathrm{mult}_b(\overline{w}_1^{-1}\overline{w}\,\overline{w}_2(c)+h\overline{w}_1^{-1}\overline{w}(\beta)-a) \\ &= \mathrm{sign}(w_1)\cdot\sum_{w_2\in{}^{(h)}W} \mathrm{sign}(w_2)\,\overline{\mathcal{N}}_{a,b}{}^{w_2(c)}\,. \end{aligned} \tag{5.10} $$

[4] In the formulation of [5, 14, 15, 16] the factor of $|\overline{W}|^{-1}$ is absent because there $\overline{\mathcal{N}}_{a,b}{}^{c}$ is taken to be non-zero only if $a,b\in\bar{P}$.

Here $\beta := \beta_2 - \overline{w}^{-1}(\beta_1)$. Together with the symmetry property (5.7) it then follows that

$$\sum_{w_3\in{}^{(h)}W} \mathrm{sign}(w_3)\, \overline{\mathcal{N}}_{w_1(a),w_2(b)}^{\quad\ w_3(c)} = \mathrm{sign}(w_1)\,\mathrm{sign}(w_2)\cdot \sum_{w_3\in{}^{(h)}W} \mathrm{sign}(w_3)\, \overline{\mathcal{N}}_{a,b}^{\ w_3(c)} \tag{5.11}$$

for all $w_1,w_2\in{}^{(h)}W$. We can rewrite this as

$$\sum_{w\in{}^{(h)}W} \epsilon_\ell(A)\epsilon_\ell(B)\,\mathrm{sign}(w)\, \overline{\mathcal{N}}_{w_A(A),w_B(B)}^{\quad\ w(c)} = \sum_{w\in{}^{(h)}W} \mathrm{sign}(w)\, \overline{\mathcal{N}}_{A,B}^{\ w(c)} \tag{5.12}$$

which by interpreting A and B as elements of $\bar{P}$ rather than ${}^{(\ell h)}P$ yields, after summation over $c\in{}^{(h)}P$,

$$\bar{f}_h(\bar\varphi_A)\star\bar{f}_h(\bar\varphi_B) = \bar{f}_h(\bar\varphi_A\star\bar\varphi_B)\,, \tag{5.13}$$

and hence shows that the maps $\bar{f}_h$ defined by (5.2) are ring homomorphisms.

Now for all $A\in{}^{(\ell h)}P$ we have $f_{\ell h,h}(\phi_A) = \epsilon_\ell(A)(\phi_{w_A(A)}) =: \epsilon_\ell(A)\cdot\varphi_a$. Then owing to (5.6) we obtain, after dividing (5.12) by $|\overline{W}|$, the relation

$${}^{(h)}\mathcal{N}_{f_{\ell h,h}(A),f_{\ell h,h}(B)}^{\quad c} = \epsilon_\ell(A)\,\epsilon_\ell(B)\,{}^{(h)}\mathcal{N}_{a,b}^{\ \ c} = \sum_{C:\ \phi_C\in f_{\ell h,h}^{-1}(\varphi_c)} \epsilon_\ell(C)\,{}^{(\ell h)}\mathcal{N}_{A,B}^{\ \ C} \tag{5.14}$$

(on the left hand side, we use the short hand notation $f_{\ell h,h}(A)\in{}^{(h)}P$ to indicate the label that corresponds to the element $\epsilon_\ell(A)f_{\ell h,h}(\phi_A)$ of ${}^{(h)}\mathcal{R}$). Summation over $c\in{}^{(h)}P$ then yields $f_{\ell h,h}(\phi_A)\star f_{\ell h,h}(\phi_B) = f_{\ell h,h}(\phi_A\star\phi_B)$, so that (5.14) is just the homomorphism property of the maps $f_{i,j}$ which were defined by (2.13) in terms of the fusion rule coefficients. (Thereby we have also obtained an alternative proof of the homomorphism property of those maps.)

To investigate further the relation between $\overline{\mathcal{R}}$ and the projective limit ${}^{(\infty)}\mathcal{R}$, we introduce the linear mappings

$$\begin{aligned} \bar\jmath_h:&\quad {}^{(h)}\mathcal{R}\to\overline{\mathcal{R}}\,, &\quad {}^{(h)}\varphi_a\mapsto\bar\varphi_a\,,\\ \jmath_{h,h'}:&\quad {}^{(h)}\mathcal{R}\to{}^{(h')}\mathcal{R}\,, &\quad {}^{(h)}\varphi_a\mapsto{}^{(h')}\varphi_a \end{aligned} \tag{5.15}$$

which map each basis element ${}^{(h)}\varphi_a$ of ${}^{(h)}\mathcal{R}$ to that basis element of $\overline{\mathcal{R}}$ and ${}^{(h')}\mathcal{R}$ ($h'\geq h$), respectively, which is labelled by the same weight $a\in{}^{(h)}P\subseteq{}^{(h')}P\subset\bar{P}$. For $h\preceq h'\preceq h''$, these maps satisfy

$$\bar\jmath_{h'}\circ\jmath_{h,h'} = \bar\jmath_h\,,\quad \jmath_{h',h''}\circ\jmath_{h,h'} = \jmath_{h,h''} \tag{5.16}$$

as well as

$$\bar{f}_h\circ\bar\jmath_h = \mathsf{id}_h\,,\quad f_{h',h}\circ\jmath_{h,h'} = \mathsf{id}_h \tag{5.17}$$

and

$$\bar{f}_h\circ\bar\jmath_{h'} = f_{h',h}\,,\quad f_{h'',h}\circ\jmath_{h',h''} = f_{h',h}\,. \tag{5.18}$$

We say that a generalized sequence ψ in the projective limit ${}^{(\infty)}\mathcal{R}$ is *ultimately constant* iff there exists a $h_\circ\in I$ such that

$$\psi(h) = \jmath_{h_\circ,h}\circ\psi(h_\circ) \tag{5.19}$$

(and hence for basis elements in particular $a_h = a_{h_\circ}$) for all heights $h\geq h_\circ$. Now assume that ψ_1 and ψ_2 are elements of ${}^{(\infty)}\mathcal{R}$ which are ultimately constant, with associated heights $h_{\circ,1}$ and $h_{\circ,2}$, respectively. Then in particular for all heights h larger than $h_\circ := 2\max(h_{\circ,1},h_{\circ,2})$ the fusion

product $\psi_1(h)\star\psi_2(h)$ in ${}^{(h)}\mathcal{R}$ is isomorphic to the product $\overline{\psi}_1\star\overline{\psi}_2$ in $\overline{\mathcal{R}}$, where $\overline{\psi}_1 := \bar{\jmath}_{h_\circ}\circ\psi_1(h_\circ)$, and analogously for $\overline{\psi}_2$. This implies that

$$(\jmath_{h_\circ,h}\circ\psi_1(h_\circ))\star((\jmath_{h_\circ,h}\circ\psi_2(h_\circ)) = \jmath_{h_\circ,h}\circ(\psi_1(h_\circ)\star\psi_2(h_\circ)) \tag{5.20}$$

even though $\jmath_{h_\circ,h}$ is not a ring homomorphism, and hence $(\psi_1\star\psi_2)(h) \equiv \psi_1(h)\star\psi_2(h) = \jmath_{h_\circ,h}\circ(\psi_1\star\psi_2)(h_\circ)$ for all $h\ge h_\circ$. Thus the product $\psi_1\star\psi_2$ is again ultimately constant. Also, the property of being ultimately constant is preserved upon taking (finite) linear transformations and conjugates. The set of ultimately constant elements therefore constitutes a subring of ${}^{(\infty)}\mathcal{R}$.

The following consideration shows that this subring is isomorphic to the fusion ring $\overline{\mathcal{R}}$. First, any ultimately constant element is a linear combination of ultimately constant elements $\psi^{(a)}$ for which $\psi^{(a)}(h_\circ)$ is an element of the canonical basis of ${}^{(h_\circ)}\mathcal{R}$, $\psi^{(a)}(h_\circ) = {}^{(h_\circ)}\varphi_a$ for some $a\in{}^{(h_\circ)}P\subset\bar{P}$. But there is a unique element ψ of ${}^{(\infty)}\mathcal{R}$ with the latter property, because at all heights h smaller than $h_\circ$ the value $\psi(h)$ is already fixed by imposing the requirement (4.6). Thus there is a bijective linear map between the subring of ultimately constant elements and the fusion ring $\overline{\mathcal{R}}$, defined by $\bar{\varphi}_a\mapsto\psi^{(a)}$ for $a\in\bar{P}$. Moreover, the same argument which led to (5.20) shows that this map is in fact an isomorphism of fusion rings. As this map is provided in a canonical manner, we can actually identify the two rings.

A generic element of ${}^{(\infty)}\mathcal{R}$ is *not* ultimately constant, so that the subring of ultimately constant elements is a proper subring of ${}^{(\infty)}\mathcal{R}$. Thus what we have achieved is to identify the fusion ring $\overline{\mathcal{R}}$ as a proper sub-fusion ring of the projective limit ring ${}^{(\infty)}\mathcal{R}$.

To conclude this section, let us remark that of course we could have enlarged by hand the category $\mathcal{F}us(\mathfrak{g})$ to a larger category $\mathcal{F}\overline{us(\mathfrak{g})}$ by just including one additional object into the category, namely the ring $\overline{\mathcal{R}}$, together with the morphisms $\bar{f}_h$. This essentially amounts to cutting the category of rings in such a way that one is able to identify the ring $\overline{\mathcal{R}}$ as the projective limit of this category $\mathcal{F}\overline{us(\mathfrak{g})}$. We do not regard this as a viable alternative to our construction, though, since when doing so one performs manipulations which are suggested merely by one's prejudice on what the limit should look like. (Also, phenomena like level-rank dualities in fusion rings require to consider various rings for different algebras $\mathfrak{g}$ on the same footing; the category $\mathcal{F}\overline{us(\mathfrak{g})}$ cannot accommodate such phenomena.) In contrast, our construction of the limit employs only the description in terms of coherent sequences, which is a natural procedure for any small category, and does not presuppose any desired features of the limit.

6 Representation theory of ${}^{(\infty)}\mathcal{R}$

A basic tool in the study of fusion rings is their representation theory. Of particular importance are the irreducible representations, which lead in particular to the notion of (generalized) quantum dimensions. In this section we show that an analogous representation theory exists for the projective limit as well. In our considerations the limit topology will again play an essential rôle.

6.1 One-dimensional representations

Let us consider for any two $h, h' \in I$ with $h' = \ell h$ the injection of the label set ${}^{(h)}P$ (defined as in (2.3)) into the label set ${}^{(h')}P$ that is defined by multiplying the weights a by a factor of ℓ:

$$a \;\mapsto\; \ell a \tag{6.1}$$

for all $a \in {}^{(h)}P$. (This induces an injection ${}^{(h)}\varphi_a \mapsto {}^{(\ell h)}\varphi_{\ell a}$ of the distinguished basis ${}^{(h)}\mathcal{B}$ of ${}^{(h)}\mathcal{R}$ into the distinguished basis ${}^{(\ell h)}\mathcal{B}$ of ${}^{(\ell h)}\mathcal{R}$. However, when this map is extended linearly to all of ${}^{(h)}\mathcal{R}$, it does *not* provide a homomorphism of fusion rings.) We can use these injections to perform an *inductive* limit of the set $({}^{(h)}P)_{h\in I}$ of label sets, where the set I (2.1) is again considered as directed via the partial ordering (2.9). We denote this inductive limit by ${}^{(\infty)}P$. An element α of ${}^{(\infty)}P$ can be characterized by an integrable weight $\alpha(h) \in {}^{(h)}P$ at some suitable height h; at any multiple ℓh of this height, the same element α of ${}^{(\infty)}P$ is then represented by the weight $\alpha(\ell h) = \ell\, \alpha(h)$. In particular, quite unlike as in the case of the projective limit, each element of the inductive limit ${}^{(\infty)}P$ is already determined by its representative at a single height. Also note that an element $\alpha \in {}^{(\infty)}P$ is *not* defined at all heights h; in particular, for any $h \in I$ the set of those $\alpha \in {}^{(\infty)}P$ which have a representative at height h is in one-to-one correspondence with the elements of ${}^{(h)}P$, and hence is in particular finite. We will use the notation $\alpha \downarrow {}^{(h)}P$ to indicate that $\alpha \in {}^{(\infty)}P$ has a representative $\alpha(h) \in {}^{(h)}P$ at height h.

We claim that any element of ${}^{(\infty)}P$ gives rise to a one-dimensional representation of the projective limit ${}^{(\infty)}\mathcal{R}$ of the fusion rings. To see this, we choose for a given $\alpha \in {}^{(\infty)}P$ a suitable height $h \in I$ such that $\alpha \downarrow {}^{(h)}P$. To any coherent sequence $(\psi(l))_{l\in I}$ in the projective limit ${}^{(\infty)}\mathcal{R}$ we then associate the number

$$\mathcal{D}_\alpha(\psi) := \frac{{}^{(h)}S_{\psi(h),\alpha(h)}}{{}^{(h)}S_{\rho,\alpha(h)}}\,, \tag{6.2}$$

i.e. the $\alpha(h)$th quantum dimension of the element $\psi(h)$ of the ring ${}^{(h)}\mathcal{R}$. Here we use the short-hand notation

$${}^{(h)}S_{\psi(h),b} := \sum_{a\in{}^{(h)}P} \zeta_a\, {}^{(h)}S_{a,b} \qquad \text{for} \;\; \psi(h) = \sum_{a\in{}^{(h)}P} \zeta_a\, {}^{(h)}\varphi_a \tag{6.3}$$

for linear combinations of S-matrix elements. Using the identities (2.18) and (2.17) as well as $\alpha(\ell h) = \ell\alpha(h)$ and the defining properties of ψ, we have

$$\frac{{}^{(\ell h)}S_{\psi(\ell h),\alpha(\ell h)}}{{}^{(\ell h)}S_{\rho,\alpha(\ell h)}} = \frac{({}^{(\ell h)}S\,D)_{\psi(\ell h),\alpha(h)}}{({}^{(\ell h)}S\,D)_{\rho,\alpha(h)}} = \frac{(F\,{}^{(h)}S)_{\psi(\ell h),\alpha(h)}}{(F\,{}^{(h)}S)_{\rho,\alpha(h)}} = \frac{{}^{(h)}S_{\psi(h),\alpha(h)}}{{}^{(h)}S_{\rho,\alpha(h)}}\,; \tag{6.4}$$

this shows that the formula (6.2) yields a well-defined map from ${}^{(\infty)}\mathcal{R}$ to $\mathbb{C}$, i.e. it does not depend on the particular choice of h. Using the knowledge about the representation theory of the rings ${}^{(h)}\mathcal{R}$, it then follows immediately that

$$\mathcal{D}_\alpha(\psi)\,\mathcal{D}_\alpha(\psi') = \sum_{a,b\in{}^{(h)}P} \zeta_a\zeta'_b\, {}^{(h)}\mathcal{N}_{a,b}{}^{c}\, \frac{{}^{(h)}S_{c,\alpha(h)}}{{}^{(h)}S_{\rho,\alpha(h)}} = \mathcal{D}_\alpha(\psi\star\psi')\,. \tag{6.5}$$

Thus the prescription (6.2) indeed provides us with a one-dimensional representation of ${}^{(\infty)}\mathcal{R}$.

Let us now associate to any element ψ of ${}^{(\infty)}\mathcal{R}$ the infinite sequence of quantum dimensions (6.2), labelled by ${}^{(\infty)}P$; this way we obtain a map

$$\mathcal{D}: \quad \psi \mapsto (\mathcal{D}_\alpha(\psi))_{\alpha\in{}^{(\infty)}P} \tag{6.6}$$

from the ring ${}^{(\infty)}\mathcal{R}$ to the algebra

$$\mathcal{X} := \{(\xi_\alpha)_{\alpha\in{}^{(\infty)}P} \,|\, \xi_\alpha \in \mathbb{C}\,\} \tag{6.7}$$

of all countably infinite sequences of complex numbers. Since we are now dealing with complex numbers rather than only integers, it is natural to consider instead of the fusion ring ${}^{(\infty)}\mathcal{R}$ the corresponding algebra over $\mathbb{C}$, to which we refer as the *fusion algebra* ${}^{(\infty)}\mathcal{A}$. (For simplicity we regard ${}^{(\infty)}\mathcal{A}$ as an algebra over $\mathbb{C}$. In principle it would be sufficient to consider it over a certain subfield of $\mathbb{C}$ generated by appropriate roots of unity.) It is then evident that the map $\mathcal{D}\colon {}^{(\infty)}\mathcal{A} \to \mathcal{X}$ defined by (6.6) is an algebra homomorphism. (We continue to use the symbol $\mathcal{D}$. More generally, below we will always assume that the various maps to be used, such as the projection (2.15), are continued $\mathbb{C}$-linearly from the fusion rings ${}^{(h)}\mathcal{R}$ to the associated fusion algebras ${}^{(h)}\mathcal{A}$, and use the same symbols for these extended maps as for the original ones.)

6.2 An isomorphism between ${}^{(\infty)}\mathcal{A}$ and $\mathcal{X}$

In this subsection we show that the map $\mathcal{D}$ introduced above even constitutes an *iso*morphism between the complex algebras ${}^{(\infty)}\mathcal{A}$ and $\mathcal{X}$:

$$\mathcal{D}: \quad {}^{(\infty)}\mathcal{A} \xrightarrow{\cong} \mathcal{X}\,. \tag{6.8}$$

Injectivity of $\mathcal{D}$ is easy to check. Suppose we have $\mathcal{D}(\psi) = 0$. Fix any $h \in I$; then all quantum dimensions of the element $\psi(h)$ of ${}^{(h)}\mathcal{R}$ vanish. From the properties of the fusion ring ${}^{(h)}\mathcal{R}$ it then follows immediately that $\psi(h) = 0$. This is true for all $h \in I$, and hence we have $\psi = 0$. This proves injectivity.

To show also surjectivity requires more work. We first need to introduce the elements

$$\mathrm{e}_a \equiv {}^{(h)}\mathrm{e}_a := {}^{(h)}S_{\rho,a} \sum_{b\in{}^{(h)}P} {}^{(h)}S^*_{a,b}\, {}^{(h)}\varphi_b \ \in {}^{(h)}\mathcal{A} \tag{6.9}$$

of the fusion algebras at height h. These elements are idempotents, i.e. obey

$$\mathrm{e}_a \star \mathrm{e}_b = \delta_{a,b}\, \mathrm{e}_a\,. \tag{6.10}$$

Owing to the unitarity of the modular transformation matrix S, the idempotents $\{{}^{(h)}\mathrm{e}_a \,|\, a\in{}^{(h)}P\}$ form a basis of the fusion algebra ${}^{(h)}\mathcal{A}$, and they constitute a partition of the unit element, in the sense that

$$\sum_{a\in{}^{(h)}P} {}^{(h)}\mathrm{e}_a = {}^{(h)}\varphi_\rho\,. \tag{6.11}$$

Also, for any element $\psi\in{}^{(\infty)}\mathcal{A}$ with $\psi(h) = \mathrm{e}_a$ and any $\alpha\in{}^{(\infty)}P$ with $\alpha{\downarrow}{}^{(h)}P$ we have

$$\mathcal{D}_\alpha(\psi) = \delta_{a,\alpha(h)}\,. \tag{6.12}$$

We now study how the idempotents $\mathrm{e}_{\alpha(h)}$ behave under the projection (2.15). First, when $\alpha\in{}^{(\infty)}\!P$ has a representative $\alpha(h)$ at height h, then for every positive integer ℓ we have, using the first of the identities (2.17),

$$\begin{aligned} f_{\ell h,h}(\mathrm{e}_{\alpha(\ell h)}) &= {}^{(\ell h)}\!S_{\rho,\alpha(\ell h)} \sum_{A\in{}^{(\ell h)}\!P} {}^{(\ell h)}\!S^*_{\alpha(\ell h),A} f_{\ell h,h}(\phi_A) = {}^{(\ell h)}\!S_{\rho,\ell\alpha(h)} \sum_{A\in{}^{(\ell h)}\!P}\sum_{b\in{}^{(h)}\!P} {}^{(\ell h)}\!S^*_{\alpha(\ell h),A} F_{A,b}\,\varphi_b \\ &= \ell^{-r/2}\cdot{}^{(h)}\!S_{\rho,\alpha(h)} \sum_{b\in{}^{(h)}\!P} ({}^{(\ell h)}\!S^*F)_{\alpha(\ell h),b}\,\varphi_b = {}^{(h)}\!S_{\rho,\alpha(h)} \sum_{b,c\in{}^{(h)}\!P} D_{\ell\alpha(h),c}\,{}^{(h)}\!S^*_{c,b}\,\varphi_b \\ &= {}^{(h)}\!S_{\rho,\alpha(h)} \sum_{b\in{}^{(h)}\!P} {}^{(h)}\!S^*_{\alpha(h),b}\,\varphi_b = \mathrm{e}_{\alpha(h)}\,. \end{aligned} \tag{6.13}$$

On the other hand, when α has a representative at height h, but not at height h', we can compute as follows. Since hh' is a multiple of h, α has a representative $\alpha(hh')$ at height hh'. Thus we can repeat the previous calculation to deduce that

$$\begin{aligned} f_{hh',h'}(\mathrm{e}_{\alpha(hh')}) &= {}^{(hh')}\!S_{\rho,\alpha(hh')}\cdot h^{r/2} \sum_{b\in{}^{(h')}\!P} (D\,{}^{(h')}\!S)_{\alpha(hh'),b}\,{}^{(h')}\!\varphi_b \\ &= (h/h')^{r/2}\cdot{}^{(h)}\!S_{\rho,\alpha(h)} \sum_{b,c\in{}^{(h')}\!P} \delta_{h'\alpha(h),hc}\,{}^{(h')}\!S_{c,b}\,{}^{(h')}\!\varphi_b\,. \end{aligned} \tag{6.14}$$

Now in the sum over c on the right hand side one has a contribution only if $c=h'\alpha(h)/h=\alpha(hh')/h$ is an element of the label set ${}^{(h')}\!P$ at height h'. But in this case we would conclude that α has in fact a representative at height h', namely $\alpha(h')=c$, which contradicts our assumption. Therefore we conclude that in the case under consideration we have $f_{hh',h'}(\mathrm{e}_{\alpha(hh')})=0$. Together with the result (6.13) it follows that by setting

$$\mathrm{e}_\alpha(h) := \begin{cases} 0 & \text{if } \alpha\in{}^{(\infty)}\!P \text{ has no representative at height } h\,, \\ \mathrm{e}_{\alpha(h)} & \text{else}\,, \end{cases} \tag{6.15}$$

we obtain an element e_α of the projective limit ${}^{(\infty)}\!\mathcal{A}$. Moreover, according to the relation (6.12) the map (6.8) acts on $\mathrm{e}_\alpha\in{}^{(\infty)}\!\mathcal{A}$ as

$$\big(\mathcal{D}(\mathrm{e}_\alpha)\big)_\beta = \delta_{\alpha,\beta} \tag{6.16}$$

for all $\alpha,\beta\in{}^{(\infty)}\!P$, and the e_α provide a partition of the unit element, analogously as in (6.11),

$$\sum_{\alpha\in{}^{(\infty)}\!P} \mathrm{e}_\alpha = \psi_\circ\,; \tag{6.17}$$

here the sum is to be understood as a limit of finite sums in the limit topology.

Now for each $h\in I$ let us define the map $g_h:\ \mathcal{X}\to{}^{(h)}\!\mathcal{A}$ by $(\xi_\alpha)_{\alpha\in{}^{(\infty)}\!P}\mapsto\sum_{\substack{\beta\in{}^{(\infty)}\!P\\ \beta\downarrow{}^{(h)}\!P}}\xi_\beta\,\mathrm{e}_{\beta(h)}$. Since $\beta\downarrow{}^{(\ell h)}\!P$ if $\beta\downarrow{}^{(h)}\!P$, we then have

$$f_{\ell h,h}\circ g_{\ell h}((\xi_\alpha)) = \sum_{\substack{\beta\in{}^{(\infty)}\!P\\ \beta\downarrow{}^{(\ell h)}\!P}} \xi_\beta\,\mathrm{e}_\beta(h) = \sum_{\substack{\beta\in{}^{(\infty)}\!P\\ \beta\downarrow{}^{(h)}\!P}} \xi_\beta\,\mathrm{e}_{\beta(h)} = g_h((\xi_\alpha)) \tag{6.18}$$

for all positive integers ℓ. Analogously we can define a map

$$g: \quad \mathcal{X} \to {}^{(\infty)}\!\mathcal{A}\,, \qquad (\xi_\alpha)_{\alpha\in{}^{(\infty)}\!P} \mapsto \sum_{\beta\in{}^{(\infty)}\!P} \xi_\beta\, \mathrm{e}_\beta \tag{6.19}$$

with similar properties. As a consequence of the relation (6.16) one finds that this map satisfies

$$\mathcal{D}\circ g = \mathsf{id}_{\mathcal{X}}\,. \tag{6.20}$$

This implies that the injective map $\mathcal{D}$ is also surjective (and that g is injective). Thus we have proven the isomorphism (6.8).

6.3 Semi-simplicity

It is known [2] that the fusion algebras ${}^{(h)}\!\mathcal{A}$ at finite heights h are semi-simple associative algebras. In this subsection we show that in a suitable topological sense the same statement holds for the projective limit ${}^{(\infty)}\!\mathcal{A}$, too.

We first combine the identity (6.10) and the definition (6.15) of the element e_α of ${}^{(\infty)}\!\mathcal{A}$ with the fact that the idempotents e_a form a basis of ${}^{(h)}\!\mathcal{A}$. This way we learn that for all $\psi\in{}^{(\infty)}\!\mathcal{A}$ and all heights $h\in I$ the fusion product $(\mathrm{e}_\alpha\star\psi)(h) = \mathrm{e}_\alpha(h)\star\psi(h)$ is proportional to $\mathrm{e}_\alpha(h)$. Thus for each $\alpha\in{}^{(\infty)}\!P$ the span

$$\mathcal{I}_\alpha := \langle \mathrm{e}_\alpha\rangle \tag{6.21}$$

of $\{\mathrm{e}_\alpha\}$ is a one-dimensional twosided ideal of the projective limit, i.e. we have ${}^{(\infty)}\!\mathcal{A}\,\mathcal{I}_\alpha = \mathcal{I}_\alpha{}^{(\infty)}\!\mathcal{A} \subseteq \mathcal{I}_\alpha$. We claim that when we endow the algebra with the limit topology, then in fact ${}^{(\infty)}\!\mathcal{A}$ is the closure of the direct sum of the ideals (6.21) in this topology:

$${}^{(\infty)}\!\mathcal{A} = \overline{\bigoplus_{\alpha\in{}^{(\infty)}\!P} \mathcal{I}_\alpha}\,. \tag{6.22}$$

(In particular, the idempotents e_α form a topological basis of ${}^{(\infty)}\!\mathcal{A}$.)

To prove this, we first recall from subsection 3.3 that in the limit topology each open set in ${}^{(\infty)}\!\mathcal{A}$ is a union of elements of the set $\Omega = \{f_h^{-1}(M)\,|\,h\in I,\ M \text{ open in } {}^{(h)}\!\mathcal{A}\}$ of all pre-images of all open sets in any of the fusion algebras ${}^{(h)}\!\mathcal{A}$. Here we assume that we have already chosen a topology on each of the fusion algebras ${}^{(h)}\!\mathcal{A}$. (Actually the choice of this topology on ${}^{(h)}\!\mathcal{A}$ will not be important; for definiteness, we may take the discrete one, as in the case of fusion rings, or also the metric topology of ${}^{(h)}\!\mathcal{A}$ as a finite-dimensional complex vector space.) Consider now an arbitrary element $\xi\in{}^{(\infty)}\!\mathcal{A}$, which because of the isomorphism ${}^{(\infty)}\!\mathcal{A}\cong\mathcal{X}$ we can write as $\xi = (\xi_\alpha)_{\alpha\in{}^{(\infty)}\!P}$. Adopting some definite numbering ${}^{(\infty)}\!P = \{\alpha_m\,|\,m\in\mathbb{N}\}$ of the countable set ${}^{(\infty)}\!P$, for $n\in\mathbb{N}$ we define

$$\hat{\xi}_n := \sum_{m\le n} \xi_{\alpha_m}\mathrm{e}_{\alpha_m} \ \in \bigoplus_{\alpha\in{}^{(\infty)}\!P} \mathcal{I}_\alpha\,. \tag{6.23}$$

To prove our assertion, we must then show that for every $h\in I$ and every open set $M\subseteq{}^{(h)}\!\mathcal{A}$ which satisfy $f_h(\xi)\in M$ we have $\hat{\xi}_n\in f_h^{-1}(M)$, i.e.

$$f_h(\hat{\xi}_n)\in M\,, \tag{6.24}$$

for all but finitely many n. Now by direct calculation we obtain

$$f_h(\hat{\xi}_n) = \sum_{\substack{m\le n \\ \alpha_m\downarrow^{(h)}P}} \xi_{\alpha_m}{}^{(h)}\mathrm{e}_{\alpha_m(h)}\,; \tag{6.25}$$

this is a finite sum, and for sufficiently large n it becomes independent of n because only finitely many $\alpha\in{}^{(\infty)}\!P$ have a representative in ${}^{(h)}\!P$. In fact, for sufficiently large n we simply have

$$f_h(\hat{\xi}_n) = \sum_{\alpha\downarrow^{(h)}P} \xi_\alpha\,{}^{(h)}\mathrm{e}_{\alpha(h)} \equiv f_h(\xi)\,. \tag{6.26}$$

Since $f_h(\xi)\in M$, this immediately shows that indeed $f_h(\hat{\xi}_n)\in M$ for almost all n, and hence the proof is completed. (Note that the fact that $f_h(\hat{\xi}_n)$ ultimately becomes equal to $f_h(\xi)$ holds for any chosen topology of ${}^{(h)}\!\mathcal{A}$, and hence the conclusion is indeed independent of that topology.)

6.4 Simple and semi-simple modules

The representation theory of ${}^{(\infty)}\!\mathcal{A}$ can now be developed by following the same steps as in the representation theory of semi-simple algebras. However, when considering modules V over ${}^{(\infty)}\!\mathcal{A}$, it is natural to restrict one's attention from the outset to *continuous* modules, i.e. to modules which are topological vector spaces and on which the representation of ${}^{(\infty)}\!\mathcal{A}$ is continuous (in particular, every element of ${}^{(\infty)}\!\mathcal{A}$ is represented by a continuous map). We will do so, and suppress the qualification 'continuous' from now on.

The one-dimensional ideals $\mathcal{I}_\alpha$ are simple modules over ${}^{(\infty)}\!\mathcal{A}$ under the (left or right) regular representation. Our first result is now that these one-dimensional modules already provide us with all simple modules, i.e. that every simple ${}^{(\infty)}\!\mathcal{A}$-module L satisfies

$$L \cong \mathcal{I}_\alpha \tag{6.27}$$

for some $\alpha\in{}^{(\infty)}\!P$.

To show this, we first observe that if $L \ncong \mathcal{I}_\alpha$, then $\mathcal{I}_\alpha L = 0$. Namely, since $\mathcal{I}_\alpha$ is an ideal of ${}^{(\infty)}\!\mathcal{A}$, we have ${}^{(\infty)}\!\mathcal{A}\,\mathcal{I}_\alpha L \subseteq \mathcal{I}_\alpha L$; thus $\mathcal{I}_\alpha L$ is a submodule of L, which by the simplicity of L implies that either $\mathcal{I}_\alpha L = L$ or $\mathcal{I}_\alpha L = 0$. In the former case, $\mathcal{I}_\alpha L = L$, we can find a vector $y\in L$ such that the space $\mathcal{I}_\alpha y$ is not zero-dimensional. Indeed, because of ${}^{(\infty)}\!\mathcal{A}\,\mathcal{I}_\alpha\, y \subseteq \mathcal{I}_\alpha\, y \subseteq L$ this space is a submodule of L, and hence by the simplicity of L it must be equal to L. It follows that the map from $\mathcal{I}_\alpha$ to L defined by $\lambda \mapsto \lambda y$ is surjective. Since L is simple, by Schur's lemma this implies that it is even an isomorphism. This shows that $L \cong \mathcal{I}_\alpha$ when $\mathcal{I}_\alpha L = L$, and hence $\mathcal{I}_\alpha L = 0$ when $L \ncong \mathcal{I}_\alpha$.
Suppose now that L is a non-zero simple module and is not isomorphic to any $\mathcal{I}_\alpha$. Then $\bigoplus_\alpha \mathcal{I}_\alpha L = 0$; since L is a continuous module, we can take the closure of this relation, so as to find that ${}^{(\infty)}\!\mathcal{A}\, L = 0$. But we have $L \subseteq {}^{(\infty)}\!\mathcal{A}\, L$, and hence this would imply that $L = 0$, which is a contradiction. Hence we learn that indeed, up to isomorphism, the ideals $\mathcal{I}_\alpha$ of ${}^{(\infty)}\!\mathcal{A}$ exhaust all the simple modules over ${}^{(\infty)}\!\mathcal{A}$.

Next we consider modules V over ${}^{(\infty)}\!\mathcal{A}$ which can be obtained from families of simple modules. Similarly as in [17, §XVII.2] one can show that the following conditions are equivalent:

(*i*) V is the closure of the sum of a family of simple submodules.

(*ii*) V is the closure of the direct sum of a family of simple submodules.

(*iii*) Every closed submodule W of V is a direct summand, i.e. there exists a closed submodule W' such that $V = W \oplus W'$.

Any (continuous) module fulfilling these equivalent conditions will be referred to as a *semi-simple* module.

The equivalence of (*i*) – (*iii*) is proven as follows. First, if $V = \overline{\sum_{i\in J} L_i}$ is the closure of a (not necessarily direct) sum of simple submodules L_i, denote by J' a maximal subset of J such that $V' := \sum_{j\in J'} L_j$ is a direct sum. Since the intersection of V' with any of the simple modules L_i is a submodule of L_i, the maximality of J' implies that $i \in J'$ and hence in fact $J' = J$. Thus (*i*) implies (*ii*).
Second, if W is a submodule of V, let J'' be the maximal subset of J such that the sum $W + \sum_{j\in J''} L_j$ is direct. Then the same arguments as before show that $V = \overline{W \oplus \bigoplus_{j\in J''} L_j}$. If, furthermore, W is closed, then it follows that $V = \overline{W} \oplus \overline{\bigoplus_{j\in J''} L_j} = W \oplus \overline{\bigoplus_{j\in J''} L_j}$. This shows that(*ii*) implies (*iii*).[5]
Third, assume that V is a non-zero module which satisfies (*iii*), and let v be a non-zero vector in V. The kernel of the homomorphism ${}^{(\infty)}\!\mathcal{A} \to {}^{(\infty)}\!\mathcal{A}\, v$ is a closed ideal of ${}^{(\infty)}\!\mathcal{A}$, which in turn is contained in a maximal closed ideal $\mathcal{J} \subset {}^{(\infty)}\!\mathcal{A}$ that is strictly contained in ${}^{(\infty)}\!\mathcal{A}$. One then has $V = \mathcal{J} v \oplus W$ and ${}^{(\infty)}\!\mathcal{A}\, v = \mathcal{J} v \oplus (W \cap {}^{(\infty)}\!\mathcal{A}\, v)$ with some submodule $W \subset V$. Now $W \cap {}^{(\infty)}\!\mathcal{A}\, v$ is simple because $\mathcal{J} v$ is maximal in ${}^{(\infty)}\!\mathcal{A}\, v$; thus V contains a simple submodule. Next let $V' \neq 0$ be the submodule of V that is the closure of the sum of all simple submodules of V. If V' were not all of V, then one would have $V = V' \oplus V''$ with $V'' \neq 0$; but by the same reasoning as before, V'' then would contain a simple submodule, in contradiction to the definition of V'. Thus $V' = V$, so we see that (*iii*) implies (*i*).

6.5 Arbitrary modules

With the characterization of semi-simple modules above, we are now in a position to study arbitrary modules of ${}^{(\infty)}\!\mathcal{A}$, in an analogous manner as in [17, §XVII.4]. Let us first assume that W is a closed submodule of a semi-simple module V, and denote by W' the closure of the direct sum of all simple submodules of W. Then there is a submodule V' of V such that $V = W' \oplus V'$. Every $w \in W$ can be uniquely written as $w = w' + v'$ with $w' \in W'$ and $v' \in V'$. Because of $v' = w - w' \in W$ we thus have $W = W' \oplus (W \cap V')$. The module $W \cap V'$ is a closed submodule of W. If it were non-zero, it would therefore (by the same reasoning as in the proof of '(*iii*) $\to$ (*i*)' in subsection 6.4) contain a simple submodule, in contradiction with the definition of W. Thus we learn that $W = W'$, or in other words:

> Every closed submodule of a semi-simple ${}^{(\infty)}\!\mathcal{A}$-module is semi-simple.

Next we consider again a closed submodule W of a semi-simple module V, and investigate the quotient module V/W. There is a closed submodule W' such that V is the direct sum $V = W \oplus W'$. Now the projection $V \to V/W$ induces a continuous isomorphism from W' to

[5] It is indeed necessary to require W to be closed. Consider e.g. the case $V = {}^{(\infty)}\!\mathcal{A}$ and $W = \bigoplus_\alpha \mathcal{I}_\alpha$. The submodule W is neither closed nor does it have a complement.

V/W. Furthermore, according to the result just obtained, W' is semi-simple. Thus we have shown:

> Every quotient module of a semi-simple ${}^{(\infty)}\!\mathcal{A}$-module with respect to a closed submodule is semi-simple.

Now any arbitrary ${}^{(\infty)}\!\mathcal{A}$-module can be regarded as a quotient module of a suitable free module modulo a closed submodule. Moreover, every free ${}^{(\infty)}\!\mathcal{A}$-module is the closure of a direct sum of countably many copies of ${}^{(\infty)}\!\mathcal{A}$ and hence is a semi-simple module. The two previous results therefore imply:

> Every ${}^{(\infty)}\!\mathcal{A}$-module is semi-simple.

Finally we consider again an arbitrary ${}^{(\infty)}\!\mathcal{A}$-module V. We denote by V_α the closure of the direct sum of all those submodules of V which are isomorphic to the simple ${}^{(\infty)}\!\mathcal{A}$-module $\mathcal{I}_\alpha$. Since each simple module over ${}^{(\infty)}\!\mathcal{A}$ is isomorphic to some $\mathcal{I}_\alpha$, any simple submodule of V is contained in V_β for some $\beta\in{}^{(\infty)}\!P$. Now every ${}^{(\infty)}\!\mathcal{A}$-module is semi-simple and hence the closure of the direct sum of its simple submodules. Thus we learn that

$$V = \overline{\bigoplus_{\alpha\in{}^{(\infty)}\!P} V_\alpha}\,. \tag{6.28}$$

Moreover, we have $\mathrm{e}_\beta V_\alpha = \delta_{\alpha,\beta} V_\alpha$ for all $\alpha,\beta\in{}^{(\infty)}\!P$, and hence

$$V_\alpha = \mathrm{e}_\alpha V = \mathcal{I}_\alpha V\,. \tag{6.29}$$

As a consequence, we see that:

> Every ${}^{(\infty)}\!\mathcal{A}$-module V can be written as
>
> $$V = \overline{\bigoplus_{\alpha\in{}^{(\infty)}\!P} \mathcal{I}_\alpha V} = \overline{\bigoplus_{\alpha\in{}^{(\infty)}\!P} \mathrm{e}_\alpha V}\,, \tag{6.30}$$
>
> and for each $\alpha\in{}^{(\infty)}\!P$ the submodule $\mathcal{I}_\alpha V$ is the closure of the direct sum of all submodules of V that are isomorphic to $\mathcal{I}_\alpha$.

We can conclude that the structure of any arbitrary (continuous) module over ${}^{(\infty)}\!\mathcal{A}$ is known explicitly, i.e. we have developed the full (topological) representation theory of ${}^{(\infty)}\!\mathcal{A}$.

6.6 Diagonalization

From the definition (6.15) of $\mathrm{e}_\alpha\in{}^{(\infty)}\!\mathcal{A}$ and the basic property (6.10) of the idempotents $\mathrm{e}_a\in{}^{(h)}\!\mathcal{A}$ it follows that the elements e_α of ${}^{(\infty)}\!\mathcal{A}$ are again idempotents:

$$\mathrm{e}_\alpha\star\mathrm{e}_\beta = \delta_{\alpha,\beta}\,\mathrm{e}_\alpha \tag{6.31}$$

for all $\alpha,\beta\in{}^{(\infty)}\!P$. In other words, by the basis transformation from the distinguished basis ${}^{(\infty)}\!\mathcal{B}$ of the fusion algebra ${}^{(\infty)}\!\mathcal{A}$ to the basis of idempotents one diagonalizes the fusion rules of ${}^{(\infty)}\!\mathcal{A}$, precisely as in the case of the algebras ${}^{(h)}\!\mathcal{A}$ at finite level.

Indeed, by combining the definitions (6.9) and (6.15) we can describe the transformation from ${}^{(\infty)}\!\mathcal{B}$ to the basis of idempotents e_α explicitly. Namely, for any $\psi=(\psi(h))_{h\in I}\in{}^{(\infty)}\!\mathcal{B}$ we have

$$\psi = \sum_{\alpha\in{}^{(\infty)}\!P} {}^{(\infty)}\!Q_{\psi,\alpha}\,\mathrm{e}_\alpha\,, \tag{6.32}$$

with

$${}^{(\infty)}\!Q_{\psi,\alpha} := \frac{{}^{(h)}\!S_{\psi(h),\alpha(h)}}{{}^{(h)}\!S_{\rho,\alpha(h)}}\,, \tag{6.33}$$

where $h\in I$ is a height at which α has a representative. Note that owing to the relation (6.4) the quotient ${}^{(\infty)}\!Q_{\psi,\alpha}$ does not depend on the particular choice of h. (This just rephrases the fact that the map $\mathcal{D}$ is an isomorphism.)

For any $\psi,\psi'\in{}^{(\infty)}\!\mathcal{B}$ we thus have

$$\psi\star\psi' = \sum_{\alpha\in{}^{(\infty)}\!P} \frac{{}^{(h)}\!S_{\psi(h),\alpha(h)}}{{}^{(h)}\!S_{\rho,\alpha(h)}}\,\frac{{}^{(h)}\!S_{\psi'(h),\alpha(h)}}{{}^{(h)}\!S_{\rho,\alpha(h)}}\,\mathrm{e}_\alpha = \sum_{\chi\in{}^{(\infty)}\!\mathcal{B}} \Big(\sum_{\alpha\in{}^{(\infty)}\!P} {}^{(\infty)}\!Q_{\psi,\alpha}\,{}^{(\infty)}\!Q_{\psi',\alpha}\,{}^{(\infty)}\!Q^-_{\alpha,\chi} \Big)\,\chi\,, \tag{6.34}$$

where ${}^{(\infty)}\!Q^-$ is the matrix for the inverse basis transformation,

$$\mathrm{e}_\alpha = \sum_{\psi\in{}^{(\infty)}\!\mathcal{B}} {}^{(\infty)}\!Q^-_{\alpha,\psi}\,\psi\,. \tag{6.35}$$

In other words, the fusion rule coefficients of the projective limit ${}^{(\infty)}\!\mathcal{A}$ can be written as

$${}^{(\infty)}\!\mathcal{N}_{\psi,\psi'}^{\ \ \psi''} = \sum_{\alpha\in{}^{(\infty)}\!P} {}^{(\infty)}\!Q_{\psi,\alpha}\,{}^{(\infty)}\!Q_{\psi',\alpha}\,{}^{(\infty)}\!Q^-_{\alpha,\psi''}\,. \tag{6.36}$$

This is nothing but the analogue of the Verlinde formula [7] that is valid for the fusion rule coefficients of the fusion algebras ${}^{(h)}\!\mathcal{A}$.

Note that already at finite height h the two indices which label the rows and columns, respectively, of the matrix ${}^{(h)}\!S$ which diagonalizes the fusion rules are a priori of a rather different nature. Namely, one of them labels the elements of the distinguished basis ${}^{(h)}\!\mathcal{B}$, while the other labels the inequivalent one-dimensional irreducible representations of ${}^{(h)}\!\mathcal{A}$. It is a quite non-trivial property of the fusion algebras which arise in rational conformal field theory (and is a prerequisite for the modularity of those fusion algebras) that nevertheless the diagonalizing matrix can be chosen such that it is symmetric, so that in particular the two kinds of labels can be treated on an equal footing [1]. Our results clearly display that this nice feature of the finite height fusion algebras ${}^{(h)}\!\mathcal{A}$ is not shared by their non-rational limit ${}^{(\infty)}\!\mathcal{A}$; in the case of ${}^{(\infty)}\!\mathcal{A}$, there seems to be no possibility to identify the two sets ${}^{(\infty)}\!\mathcal{B}$ and ${}^{(\infty)}\!P$ which label the elements of the distinguished basis and the one-dimensional irreducible representations, respectively, with each other.

On the other hand, our results show that the projective limit ${}^{(\infty)}\!\mathcal{R}$ that we constructed in this paper still possesses all those structural properties of a modular fusion ring which can reasonably be expected to survive in the limit of infinite level.

Acknowledgement. We are grateful to I. Kausz and B. Pareigis for helpful comments.

References

[1] J. Fuchs, *Fusion rules in conformal field theory*, Fortschr. Phys. 42 (1994) 1

[2] T. Kawai, *On the structure of fusion algebras*, Phys. Lett. B 287 (1989) 247

[3] A. Ashtekar and J. Lewandowski, *Projective techniques and functional integration for gauge theories*, J. Math. Phys. 36 (1995) 2170

[4] G. Bimonte, E. Ercolessi, G. Landi, F. Lizzi, G. Sparano, and P. Teotonio-Sobrinho, *Lattices and their continuum limits*, preprint hep-th/9507147

[5] V.G. Kac, *Infinite-dimensional Lie Algebras*, third edition (Cambridge University Press, Cambridge 1990)

[6] J. Fuchs, A.N. Schellekens, and C. Schweigert, *Quasi-Galois symmetries of the modular S-matrix*, Commun. Math. Phys. 176 (1996) 447

[7] E. Verlinde, *Fusion rules and modular transformations in 2D conformal field theory*, Nucl. Phys. B 300 [FS22] (1988) 360

[8] *Encyclopaedia of Mathematics* (Kluwer Academic Publishers, Dordrecht 1991), p. 333

[9] M. Artin, *Grothendieck Topologies* (Harvard University, Boston 1962; unpublished script)

[10] B. Pareigis, *Categories and Functors* (Academic Press, New York 1970)

[11] P.J. Hilton and U. Stammbach, *A Course in Homological Algebra* (Springer Verlag, New York 1970)

[12] G. Racah, in: Istanbul Summer School on *Group Theoretical Concepts and Methods in Elementary Particle Physics*, F. Gursey, ed. (Gordon and Breach, New York 1964), p. 1

[13] D. Speiser, *Fundamental representations of Lie groups*, Helv. Phys. Acta 38 (1965) 73

[14] M. Walton, *Fusion rules in Wess–Zumino–Witten models*, Nucl. Phys. B 340 (1990) 777

[15] P. Furlan, A.Ch. Ganchev, and V.B. Petkova, *Quantum groups and fusion rules multiplicities*, Nucl. Phys. B 343 (1990) 205

[16] J. Fuchs and P. van Driel, *WZW fusion rules, quantum groups, and the modular matrix S*, Nucl. Phys. B 346 (1990) 632

[17] S. Lang, *Algebra* (Addison–Wesley, Reading 1984)